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WHITNEY-TYPE EXTENSIONS IN QUASI-METRIC SPACES

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Abstract. We discuss geometrical scenarios guaranteeing that functions defined on a given set may be extended to the entire ambient, with preservation of the class of regularity. This extends to arbitrary quasi-metric spaces work done by E.J. McShane in the context of metric spaces, and to geometrically doubling quasi-metric spaces work done by H. Whitney in the Euclidean setting. These generalizations are quantitatively sharp.

1. Introduction.

One important problem in analysis which has received a considerable amount of attention over the years pertains to the issue of extending classes of functions satisfying certain regularity properties (e.g., continuity, Lipschitzianity) from a subset of an ambient to the entire space while retaining the regularity properties in question. Henceforth, this generic question will be referred to as the *extension problem* (with the tacit agreement that this is allowed to acquire various concrete nuances in subsequent re-formulations). For example, if E is a closed subset of a metric space (X, d), then Hausdorff's formula (cf., e.g., [7, Exercise 4.1.F])

$$F(x) := \begin{cases} \inf\Big\{f(y) + \frac{d(x,y)}{\mathrm{dist}_d(x,E)} - 1 : y \in E\Big\} & \text{if } x \in X \setminus E, \\ f(x) & \text{if } x \in E, \end{cases} \tag{1}$$

gives an extension of a given continuous function $f : E \to \mathbb{R}$ to the entire ambient space which is continuous on X (above, $\mathrm{dist}_d(x, E) := \inf\{d(x, y) : y \in E\}$ for each $x \in X$). This may be thought of as an explicit version of Tietze's extension theorem (classically formulated in normal topological spaces; cf., e.g., [24, Theorem 35.1, p. 219]) in the setting of metric spaces. In relation to Hausdorff's formula, it is also significant to note that the assignment $f \mapsto F$ described in (1) is nonlinear.

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Regarding the extension of functions with preservation of higher regularity we wish to mention the pioneering work of E.J. McShane [22], H. Whitney [32] and M.D. Kirszbraun [17]. To state the main result in [22] we shall need some notation. Given a metric space (X, d), denote by $\mathrm{Lip}(X, d)$ the vector space of real-valued functions defined on X which do not increase distances by more than a fixed multiplicative factor, i.e.,

$$\mathrm{Lip}(X, d) := \Big\{ f : X \to \mathbb{R} : \|f\|_{\mathrm{Lip}(X,d)} := \sup_{x,y \in X, x \neq y} \tfrac{|f(x)-f(y)|}{d(x,y)} < +\infty \Big\}. \tag{2}$$

The question concerning McShane's work in [22] is whether given a metric space (X, d), a nonempty set $E \subseteq X$, and a Lipschitz function $f \in \mathrm{Lip}(E, d)$ (regarding E, equipped with the restriction of d to $E \times E$ as a metric space in its own right), it is possible to find $F \in \mathrm{Lip}(X, d)$ such that

$$F\big|_E = f \quad \text{ and } \quad \|F\|_{\mathrm{Lip}(X,d)} = \|f\|_{\mathrm{Lip}(E,d)}. \tag{3}$$

McShane's elegant solution is based on the observation that either

$$f^*(x) := \sup\big\{ f(y) - \|f\|_{\mathrm{Lip}(E,d)} d(x, y) : y \in E \big\}, \qquad \forall x \in X, \tag{4}$$

or

$$f_*(x) := \inf\big\{ f(y) + \|f\|_{\mathrm{Lip}(E,d)} d(x, y) : y \in E \big\}, \qquad \forall x \in X, \tag{5}$$

are such extensions for the given function f. In fact, the upper and lower McShane extensions constructed in (4) and (5) are extremal in the following sense: if $F \in \mathrm{Lip}(X, d)$ is a function with the property that $F|_E = f$ and $\|F\|_{\mathrm{Lip}(X,d)} = \|f\|_{\mathrm{Lip}(E,d)}$, then necessarily $f^* \leq F \leq f_*$ on X.

While McShane's extension theorem just described has the distinct attribute that it works in the general setting of arbitrary metric spaces, as with (1) before, the extension operators $f \mapsto f^*$ and $f \mapsto f_*$ are nonlinear. Furthermore, McShane's original argument in [22] does not yield an extension satisfying (3) in the more general case when the Lipschitz functions in question are vector-valued. The latter scenario is considerably more subtle and a remarkable positive result in this regard has been established by M.D. Kirszbraun in 1934. Specifically, Kirszbraun's theorem (cf. [17, p. 104, Hauptsatz I]) asserts that if E is a subset of $\mathbb{R}^n$ then any vector-valued Lipschitz function $f : E \to \mathbb{R}^m$ may be extended to a Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}^m$ which has the same Lipschitz constant as f. A further generalization of this result to the case when the Euclidean spaces $\mathbb{R}^n$, $\mathbb{R}^m$ are replaced by two arbitrary Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$ may be found in [28, p. 21], via a proof which utilizes Hausdorff's maximal principle as well as geometric characteristics of Hilbert spaces. This is relevant since the corresponding statement for Banach space-valued Lipschitz functions is not true in general, even for finite-dimensional Banach spaces (cf., e.g., [28, p. 20] for a discussion which shows that the preservation of the Lipschitz constant fails even in such simple cases as $\mathbb{R}^m$ equipped with some ℓ_p norm with $p \neq 2$; see also [10, p. 202] for a similar discussion involving the ℓ_∞ norm). Kirszbraun's theorem was subsequently reproved by F.A. Valentine in [30], [31] (and, for this matter, it is occasionally referred to as the Kirszbraun-Valentine theorem).

In the same year, 1934, in which the papers [22], [17] of McShane and Kirszbraun have appeared, H. Whitney has published a rather influential article, [32], dealing with the extension problem. Based on a different circle of ideas, Whitney succeeds in constructing an extension operator in the Euclidean setting which is both linear and preserves higher degrees of smoothness. Somewhat more specifically, in [32], Whitney gave necessary and sufficient conditions on an array of functions $\{f^\alpha\}_{|\alpha|\leq m}$ defined on a closed subset E of $\mathbb{R}^n$ ensuring the existence of a function $F \in \mathscr{C}^m(\mathbb{R}^n)$ with the property that $(\partial^\alpha F)|_E = f^\alpha$ whenever the multi-index α satisfies $|\alpha| \leq m$. In addition, Whitney's extension operator $\{f^\alpha\}_{|\alpha|\leq m} \mapsto F$ is universal (in the sense that it simultaneously preserves all orders of smoothness), as well as linear. A timely exposition of this result may be found in E.M. Stein's monograph [29, pp. 170–180]. In particular, the proof of [29, Theorem 3, p. 174] dealing with the extension problem in the class of real-valued Hölder (and Lipschitz) continuous functions defined in open subsets of $\mathbb{R}^n$ makes use of three basic ingredients, namely:

- [i] the existence of a Whitney decomposition of an open subset of $\mathbb{R}^n$ (into Whitney balls of bounded overlap),
- [ii] the existence of a $\mathscr{C}^\infty$ smooth partition of unity subordinate (in an appropriate, quantitative manner) to such a decomposition, and
- [iii] differential calculus in open subsets of $\mathbb{R}^n$ along with other specific structural properties of the Euclidean space.

Extension theorems of the type discussed above are useful for a tantalizing array of purposes. On the theoretical side, such results constitute a versatile, powerful tool for dealing with problems in the areas of Harmonic Analysis and Partial Differential Equations, (cf., e.g., the discussion in [12], [14] [19], [29], as well as in the references cited there), while on the practical side they have found to be useful in a variety of areas of Applied Mathematics (cf., e.g., [3], [4], [27] for applications to image processing). The work initiated by McShane and Whitney in the 1930's continues to exert a significant degree of influence, and the extension problem continues to be an active area of research. For example, the monograph [15] by A. Jonsson and H. Wallin is devoted to establishing Whitney-type extension results for (arrays of) functions defined on closed subsets of $\mathbb{R}^n$ whose smoothness is measured on Besov and Triebel-Lizorkin scales (intrinsically defined on those closed sets). Also, in [2], Y. Brudnyi and P. Shvartsman have produced intrinsic characterizations of the restrictions to a given closed subset of $\mathbb{R}^n$ of functions from $\mathscr{C}^{1,\omega}(\mathbb{R}^n)$ and, building on this work, in a series of papers (cf. [8]-[9] and the references therein) C. Fefferman has further developed this circle of ideas by producing certain sharp versions of Whitney's extension result in the higher order smoothness case.

In contrast with C. Fefferman's work just mentioned, which deals with preservation of higher smoothness ($\mathscr{C}^{k,\omega}$ with $k \geq 1$, i.e., functions whose partial derivatives exist up to order k and have modulus of continuity ω) in the Euclidean setting, our goal here is the study of the extension problem at the low end of the spectrum of smoothness (corresponding to $\mathscr{C}^{k,\omega}$ in which one takes $k = 0$), and with the Euclidean ambient replaced by an abstract quasi-metric space (X, ρ). In this setting, the class of smoothness we wish to preserve under extension is denoted by $\dot{\mathscr{C}}^\omega$ and generalizes the class of Lipschitz functions in metric spaces. More specifically, functions in $\dot{\mathscr{C}}^\omega$ have moduli of continuity controlled in terms of a fixed mapping $\omega : [0, +\infty) \to [0, +\infty)$ assumed (among other things) to be β-subadditive, for some finite parameter β related to the degree to which ρ fails to be a genuine distance on X, that is,

$$0 < \beta \leq \left(\log_2 \Big[\sup_{\substack{x,y,z \in X \\ \text{not all equal}}} \frac{\rho(x,y)}{\max\{\rho(x,z), \rho(z,y)\}} \Big] \right)^{-1}. \tag{6}$$

The reader is referred to Definition 3.1 as well as to (40)-(41) in the body of the paper for more details. Here we only wish to note that in the case when the quasi-metric space (X,ρ) is a genuine metric space (as in McShane's work in [22]), it follows from (6) that $\beta = 1$ is an admissible value which, in turn, allows us to consider $\omega(t) := ct$ for all $t \geq 0$ (with $c \in (0,+\infty)$ fixed). This is significant since $\dot{\mathscr{C}}^\omega$ becomes the class of Lipschitz functions precisely for such choices of ω.

En route to the main result of the paper (reviewed below), we shall need to first generalize McShane's approach from metric spaces and Lipschitz functions to general quasi-metric spaces and functions in the class $\dot{\mathscr{C}}^\omega$. In a version devoid of technical jargon, this theorem reads as follows (see also Theorem 3.3 in the body of the paper):

Theorem 1.1. *Let (X,ρ) be a quasi-metric space and let $\omega : [0,+\infty) \to [0,+\infty)$ be such that*

$$\begin{array}{l} \omega \textit{ is non-decreasing on } [0,+\infty), \\ \omega(t) > 0 \textit{ for all } t > 0, \omega(0) = 0, \\ \exists \beta, \textit{ finite, so that (6) holds and} \\ \omega\big((s^\beta + t^\beta)^{1/\beta}\big) \leq \omega(s) + \omega(t) \textit{ for all } s, t \geq 0. \end{array} \tag{7}$$

Then there exists a finite constant $C = C(\rho,\beta) > 0$ with the property that if $E \subseteq X$ is a set of cardinality ≥ 2 and if $f : E \to \mathbb{R}$ is a given function, then there exists $F : X \to \mathbb{R}$ such that $f = F\big|_E$ and

$$\sup_{x,y\in X, x\neq y} \frac{|F(x)-F(y)|}{\omega(\rho(x,y))} \leq C \sup_{x,y\in E, x\neq y} \frac{|f(x)-f(y)|}{\omega(\rho(x,y))}. \tag{8}$$

The main result in this paper, discussed in Theorem 1.2 below, provides a solution to the extension problem via the construction of a bounded linear extension operator $\mathscr{E} : f \mapsto F$ which preserves the class $\dot{\mathscr{C}}^\omega$. In contrast to the work of Whitney in [32] carried out in the Euclidean setting, this is accomplished by working in a quasi-metric space (X,ρ) assumed to be *geometrically doubling*. The latter condition, which may me thought of as a quantitative, scale-invariant version of the fact that ρ-balls are totally bounded[1], amounts to the ability of covering any given ρ-ball by at most a fixed number of ρ-balls twice as small as the original one (cf. Definition 6.1). More specifically, in such a context we prove the following extension theorem:

Theorem 1.2. *Let (X,ρ) be a geometrically doubling quasi-metric space, fix a closed subset E of X of cardinality at least two, and suppose that the function $\omega : [0,+\infty) \to [0,+\infty)$ is as in (7). In addition, suppose that $(\mathscr{V}, \|\cdot\|_{\mathscr{V}})$ is a quasi-normed vector space. Then there exists an operator $\mathscr{E}$ mapping the vector space of $\mathscr{V}$-valued functions defined on E into the vector space of $\mathscr{V}$-valued functions defined on X linearly and such that*

$$\sup_{x,y\in X, x\neq y} \frac{\|(\mathscr{E}f)(x)-(\mathscr{E}f)(y)\|_{\mathscr{V}}}{\omega(\rho(x,y))} \leq C \sup_{x,y\in E, x\neq y} \frac{\|f(x)-f(y)\|_{\mathscr{V}}}{\omega(\rho(x,y))}, \tag{9}$$

for any function $f : E \to \mathscr{V}$, where $C = C(\rho,\beta,\mathscr{V}) \in (0,+\infty)$ is a constant independent of f.

The reader is referred to Theorem 7.1 in the body of the paper for a somewhat more refined and informative formulation of this result.

[1]A subset E of a quasi-metric space (X,ρ) is called *totally bounded* provided for any $r > 0$ there exists a covering of E by a finite family of ρ-balls of radii r.

The next two paragraphs contain some comments about the relevance and sharpness of Theorem 1.1 and Theorem 1.2. The fact that Theorems 1.1-1.2 are formulated in the setting of quasi-metric spaces is rather significant for applications. In this vein, it is worth recalling that the most natural setting in which the bulk of the Calderón-Zygmund theory of singular integral operators may be developed is that of spaces of homogeneous type[2]. However, as opposed to the setting of metric spaces (considered in [17], [22], [32]) where Lipschitz functions are abundant[3], the space of Lipschitz functions is often trivial (i.e., it reduces to just constants) in the framework of quasi-metric spaces. For example, elementary calculus shows that this is the case for (X_o, ρ_o) if, for some fixed $\gamma > 1$, we take

$$\begin{gathered} X_o \text{ a nonempty, connected, open subset of } \mathbb{R}^n \\ \text{and } \rho_o(x,y) := |x-y|^\gamma \text{ for every } x, y \in X_o. \end{gathered} \tag{10}$$

In turn, the fact that the only functions which are globally Lipschitz on such a quasi-metric space (X_o, ρ_o) are constant functions shows that the extension problem of Lipschitz functions fails to have a solution in this setting. Indeed, if the cardinality of $E \subseteq X_o$ is finite and ≥ 2, then any non-constant real-valued function f defined on E is Lipschitz but does not extend to a Lipschitz function F on (X_o, ρ_o), since the latter would have to be constant.

The moral of this discussion is that once the focus shifts from metric spaces to the more general category of quasi-metric spaces, one necessarily has to formulate the extension problem for classes of functions other than Lipschitz. Given such a quasi-metric space (X, ρ), one natural replacement of $\mathrm{Lip}(X, \rho)$ is the class of Hölder functions of order $\beta \in (0, +\infty)$,

$$\dot{\mathscr{C}}^\beta(X,\rho) := \Big\{ f : X \to \mathbb{R} : \sup_{x,y \in X, x \neq y} \tfrac{|f(x)-f(y)|}{\rho(x,y)^\beta} < +\infty \Big\}, \tag{11}$$

and our space $\dot{\mathscr{C}}^\omega$ reduces precisely to (11) when $\omega(t) := t^\beta$. Then, if β is as in (6) it follows that this ω satisfies all conditions in (7), hence *Theorems 1.1-1.2 work for the Hölder class* (11) *granted* (6). Remarkably, this corollary is sharp. To see this, for some fixed $\gamma > 0$ consider (X_o, ρ_o) as in (10) and note that, in this scenario, (6) becomes $0 < \beta \leq \gamma^{-1}$. On the other hand, if $\beta > \gamma^{-1}$ then, since Hölder functions of order β with respect to ρ_o are Hölder functions of order $\beta\gamma > 1$ with respect to the standard Euclidean distance in the nonempty, open, connected subset X_o of $\mathbb{R}^n$, it follows that this Hölder class contains only constant functions. Hence, much as in the discussion following (10), the extension problem does not have a solution in this setting.

To the best of our knowledge, this is the first time the extension problem (as addressed in Theorems 1.1-1.2) has been considered in the setting of quasi-metric spaces. While the strategy for dealing with Theorem 1.1 is related to that employed in [22] where metric spaces (and Lipschitz functions) have been considered, the geometry of quasi-metric spaces can be significantly more intricate. To cope with this aspect, we employ a sharp metrization theorem (cf. Theorem 2.1) recently established in [23] which, in turn, extends work by R.A. Macías and C. Segovia in [20]. Among other things, given a quasi-metric space (X, ρ) this allows us to identify the optimal range of exponents β with the property that ρ^β is pointwise equivalent to a metric on X.

[2] A space of homogeneous type is a quasi-metric space equipped with a doubling measure.

[3] Trivially, if (X, d) is a metric space then for each fixed $x_o \in X$ the function $d(\cdot, x_o) : X \to \mathbb{R}$ is Lipschitz.

Our strategy for dealing with Theorem 1.2 makes essential use of Theorem 1.1. More specifically, we use Theorem 1.1 in order to prove a quantitative Urysohn's lemma, stated as Theorem 4.1. Recall that the classical Urysohn's lemma is a separation result of a purely topological nature, involving continuous functions. The point of Theorem 4.1 is that, in contrast with the classical setting, if the ambient is a quasi-metric space then the continuity aspects of the separation phenomenon alluded to above may be quantified (in terms of the quasi-distance between the two sets being separated).

In turn, the quantitative Urysohn lemma presented in Theorem 4.1 is used to produce a partition of unity $(\varphi_j)_j$ subordinate to a Whitney-type decomposition (as discussed in Theorem 6.2) of an open set in a geometrically doubling quasi-metric space, which consists of bump-functions belonging to the class $\dot{\mathscr{C}}^\omega$ and whose normalization exhibits natural scaling-like properties. Roughly speaking, the above scaling property regards the correlation between the $\dot{\mathscr{C}}^\omega$ semi-norms of the bump functions $(\varphi_j)_j$ to the magnitude of the separation between the extreme level sets $\varphi_j^{-1}(\{0\})$ and $\varphi_j^{-1}(\{1\})$, in a dilation invariant-like fashion. A precise formulation may be found in Theorem 5.1.

Once these ingredients are in place, the incisive step in the proof of Theorem 1.2 is the actual set-up of a Whitney-type extension operator, based upon the Whitney decomposition and Whitney-like partition of unity results just described. More specifically, given a closed subset E of a geometrically doubling quasi-metric space X, the extension operator we consider is

$$(\mathscr{E}f)(x) := \begin{cases} f(x) & \text{if } x \in E, \\ \sum\limits_j f(p_j)\varphi_j(x) & \text{if } x \in X \setminus E, \end{cases} \qquad \forall x \in X, \tag{12}$$

where f is an arbitrary $\mathscr{V}$-valued function defined on E, $(\varphi_j)_j$ is a partition of unity of the type described above, subordinate to a Whitney-type decomposition of the open set $X \setminus E$ into ρ-balls $\{B_j\}_j$, and $p_j \in E$ is the "nearest" point in E to B_j. Although the proof of Theorem 1.2 retains, in a broad sense, the strategy presented in [29], the execution is necessarily different given the minimality of the structures involved in the setting we are considering here (compare with [i]-[iii] listed earlier). While this degree of generality is certainly desirable given the large spectrum of applications of such a result, it is interesting to note that the absence of miracles associated with differentiability, vector space structure, and Euclidean geometry actually better elucidates the nature of the phenomenon at hand.

The layout of this paper is as follows. In Section 2 we review basic terminology and results pertaining to quasi-metric spaces and classes of functions measuring smoothness. In particular, here we record a sharp metrization theorem, recently established in [23] extending earlier work in [20]. The main result in Section 3 is Theorem 1.1, generalizing [22, Theorem 1, p. 838]. Section 4 deals with topological separation properties by means of functions from the class $\dot{\mathscr{C}}^\omega$ and the main result here, Theorem 4.1, may be regarded as a quantitative version of the classical Urysohn lemma. In Section 5 we prove the existence of Whitney-like partitions of unity as described in Theorem 5.1. Moving on, in Section 6 we discuss a Whitney-type decomposition result which extends work by R. Coifman and G. Weiss in [5, Theorem 3.1, p. 71] and [6, Theorem 3.2, p. 623] (see Theorem 6.2). Finally, in Section 7, we formulate and prove our principal result in this paper, Theorem 7.1, which is an extension result akin Whitney's work in the Euclidean setting, formulated in the context of geometrically doubling quasi-metric spaces.

In closing, we wish to emphasize that our generalization of McShane and Whitney's results is done under minimal structural assumptions and without compromising the quantitative aspects of the results in question. In addition, a significant number of preliminary results proved here (e.g., a quantitative Urysohn lemma, Whitney-like partitions of unity) are of independent interest and should be useful for other problems in the areas of analysis on quasi-metric spaces.

2. **Classes of functions quantifying continuity on quasi-metric spaces.** In this section we review the notion of quasi-metric space (along with related metric and topological matters), discuss a recent sharp metrization theorem proved in [23], and introduce certain classes of functions on quasi-metric spaces whose regularity is suitably quantified.

To get started, given a nonempty set X, call a function $\rho : X \times X \to [0, +\infty)$ a `quasi-distance` provided there exist two finite constants $C_0, C_1 \geq 1$ with the property that for every $x, y, z \in X$, one has

$$\begin{gathered} \rho(x,y) = 0 \Longleftrightarrow x = y, \\ \rho(y,x) \leq C_0 \rho(x,y) \quad \text{ and} \\ \rho(x,y) \leq C_1 \max\{\rho(x,z), \rho(z,y)\}. \end{gathered} \tag{13}$$

In the sequel, we shall denote by $\mathfrak{Q}(X)$ the collection of all quasi-distances on X. Going further, call two functions $\rho_1, \rho_2 : X \times X \to [0, +\infty)$ `equivalent`, and write $\rho_1 \approx \rho_2$, if there exist $C', C'' \in (0, +\infty)$ with the property that

$$C'\rho_1 \leq \rho_2 \leq C''\rho_1 \quad \text{on } X \times X. \tag{14}$$

It is then clear that if $\rho \in \mathfrak{Q}(X)$ and $\rho' : X \times X \to [0, +\infty)$ is such that $\rho' \approx \rho$ then $\rho' \in \mathfrak{Q}(X)$ as well.

For each $\rho \in \mathfrak{Q}(X)$ we define C_ρ to be the smallest constant which can play the role of C_1 in the last inequality in (13), i.e.,

$$C_\rho := \sup_{\substack{x,y,z \in X \\ \text{not all equal}}} \frac{\rho(x,y)}{\max\{\rho(x,z), \rho(z,y)\}} \in [1, +\infty), \tag{15}$$

and define $\widetilde{C}_\rho$ to be the smallest constant which can play the role of C_0 in the first inequality in (13), i.e.,

$$\widetilde{C}_\rho := \sup_{\substack{x,y \in X \\ x \neq y}} \frac{\rho(y,x)}{\rho(x,y)} \in [1, +\infty). \tag{16}$$

By a `quasi-metric space` we shall understand a pair (X, ρ) where X is a set of cardinality ≥ 2, and ρ is a quasi-distance on X. Given a quasi-metric space (X, ρ) define the ρ-`ball` centered at $x \in X$ with radius $r > 0$ to be

$$B_\rho(x, r) := \{y \in X : \rho(x,y) < r\}. \tag{17}$$

Also, call $E \subseteq X$ `bounded` if E is contained in a ρ-ball, and define its diameter as

$$\operatorname{diam}_\rho(E) := \sup\big\{\rho(x,y) : x, y \in E\big\}. \tag{18}$$

The ρ-`distance` between two arbitrary, nonempty sets $E, F \subseteq X$ is naturally defined as

$$\operatorname{dist}_\rho(E, F) := \inf\{\rho(x,y) : x \in E, y \in F\}, \tag{19}$$

and if the set $E = \{x\}$ for some point $x \in X$ and $F \subseteq X$, we shall abbreviate $\operatorname{dist}_\rho(x, F) := \operatorname{dist}_\rho(\{x\}, F)$.

Turning to topological considerations, we note that any quasi-metric space (X,ρ) has a canonical topology, naturally induced by the quasi-distance ρ which we will denote by τ_ρ. The latter is defined as the largest topology on X with the property that for each point $x \in X$ the family $\{B_\rho(x,r)\}_{r>0}$ is a fundamental system of neighborhoods of x. In concrete terms,

$$\mathcal{O} \in \tau_\rho \overset{\text{def}}{\Longleftrightarrow} \mathcal{O} \subseteq X \text{ and } \forall\, x \in \mathcal{O}\ \exists\, r > 0 \text{ such that } B_\rho(x,r) \subseteq \mathcal{O}. \tag{20}$$

As is well-known, the topology induced by the given quasi-distance on a quasi-metric space is metrizable. Below we shall review a recent result proved in [23] which is a sharp quantitative version of this fact. To facilitate the subsequent discussion we first make a couple of definitions. Assume that X is an arbitrary, nonempty set. Given an arbitrary function $\rho : X \times X \to [0,+\infty]$ and an arbitrary exponent $\alpha \in (0,+\infty]$ define the function

$$\rho_\alpha : X \times X \longrightarrow [0,+\infty] \tag{21}$$

by setting for each $x, y \in X$

$$\begin{aligned}\rho_\alpha(x,y) := \inf\Big\{\Big(\sum_{i=1}^{N} \rho(\xi_i,\xi_{i+1})^\alpha\Big)^{\frac{1}{\alpha}} : N \in \mathbb{N} \text{ and } \xi_1,\dots,\xi_{N+1} \in X, \qquad (22)\\ \text{(not necessarily distinct) such that } \xi_1 = x \text{ and } \xi_{N+1} = y\Big\},\end{aligned}$$

whenever $\alpha < +\infty$, and its natural counterpart corresponding to the case when $\alpha = +\infty$, i.e.,

$$\begin{aligned}\rho_\infty(x,y) := \inf\Big\{\max_{1\le i\le N} \rho(\xi_i,\xi_{i+1}) : N \in \mathbb{N} \text{ and } \xi_1,\dots,\xi_{N+1} \in X, \qquad (23)\\ \text{(not necessarily distinct) such that } \xi_1 = x \text{ and } \xi_{N+1} = y\Big\}.\end{aligned}$$

It is then clear from definitions that

$$\forall \rho \in \mathfrak{Q}(X), \forall \alpha \in (0,+\infty] \Longrightarrow \rho_\alpha \in \mathfrak{Q}(X) \text{ and } \rho_\alpha \le \rho \text{ on } X. \tag{24}$$

Going further, if $\rho : X \times X \to [0,+\infty]$ is an arbitrary function, consider its symmetrization ρ_{sym} defined by

$$\rho_{sym} : X \times X \longrightarrow [0,+\infty], \quad \rho_{sym}(x,y) := \max\{\rho(x,y),\rho(y,x)\}, \quad \forall x,y \in X. \tag{25}$$

Then ρ_{sym} is symmetric, i.e., $\rho_{sym}(x,y) = \rho_{sym}(y,x)$ for every $x, y \in X$, and $\rho_{sym} \ge \rho$ on $X \times X$. In fact, ρ_{sym} is the smallest $[0,+\infty]$-valued function defined on $X \times X$ which is symmetric and pointwise $\ge \rho$. Furthermore, if ρ is as in (13) then

$$\rho_{sym} \in \mathfrak{Q}(X), C_{\rho_{sym}} \le C_\rho, \widetilde{C}_{\rho_{sym}} = 1, \text{ and } \rho \le \rho_{sym} \le \widetilde{C}_\rho \rho. \tag{26}$$

Here is the quantitative metrization theorem from [23] alluded to above.

Theorem 2.1. *Let (X,ρ) be a quasi-metric space and assume that $C_\rho, \widetilde{C}_\rho \in [1,+\infty)$ are as in* (15)-(16). *In this context, define* (*cf.* (22)-(23))

$$\rho_\# := (\rho_{sym})_\alpha \text{ for } \alpha := (\log_2 C_\rho)^{-1} \in (0,+\infty]. \tag{27}$$

Then

$$\rho_\# : X \times X \longrightarrow [0,+\infty) \quad \text{is continuous,} \tag{28}$$

when $X \times X$ is equipped with the natural product topology $\tau_\rho \times \tau_\rho$. Moreover, for any finite number $\beta \in (0, \alpha]$, the function

$$d_{\rho,\beta} : X \times X \to [0, +\infty), \qquad d_{\rho,\beta}(x, y) := \big[\rho_\#(x, y)\big]^\beta, \qquad \forall x, y \in X, \tag{29}$$

is a distance on X, i.e. for every $x, y, z \in X$, $d_{\rho,\beta}$ satisfies

$$d_{\rho,\beta}(x, y) = 0 \iff x = y \tag{30}$$

$$d_{\rho,\beta}(x, y) = d_{\rho,\beta}(y, x) \tag{31}$$

$$d_{\rho,\beta}(x, y) \leq d_{\rho,\beta}(x, z) + d_{\rho,\beta}(z, y), \tag{32}$$

and has the property that $(d_{\rho,\beta})^{1/\beta} \approx \rho$. More specifically,

$$(C_\rho)^{-2}\rho(x, y) \leq \big[d_{\rho,\beta}(x, y)\big]^{1/\beta} = \rho_\#(x, y) \leq \widetilde{C}_\rho \rho(x, y), \quad \forall x, y \in X. \tag{33}$$

In particular, the topology induced by the distance $d_{\rho,\beta}$ on X is precisely τ_ρ.

Remark 1. In the context of Theorem 2.1, it has also been shown in [23] that $\rho_\#$ satisfies the following Hölder-type regularity condition of order β (having the same significance as before):

$$\begin{aligned} \big|\rho_\#(x, y) - \rho_\#(z, w)\big| \quad &\leq \quad \tfrac{1}{\beta} \max\big\{\rho_\#(x, y)^{1-\beta}, \rho_\#(z, w)^{1-\beta}\big\} \\ &\qquad \times \big(\big[\rho_\#(x, z)\big]^\beta + \big[\rho_\#(y, w)\big]^\beta\big) \end{aligned} \tag{34}$$

whenever $x, y, z, w \in X$ (with the understanding that when $\beta \geq 1$ one also imposes the conditions $x \neq y$ and $z \neq w$). This is, of course, a stronger property than (28).

The key feature of the result discussed in Theorem 2.1 is the fact that if (X, ρ) is any quasi-metric space then ρ^β is equivalent to a distance on X for any finite number $\beta \in (0, (\log_2 C_\rho)^{-1}]$. This result is sharp and improves upon an earlier version due to R.A. Macías and C. Segovia [20], in which these authors have identified a non-optimal upper-bound for the exponent β.

In the second part of this section we elaborate on certain classes of functions measuring smoothness on quasi-metric spaces which are relevant for the present work. Concretely, let (X, ρ) be a reference quasi-metric space, and assume that $(\mathscr{V}, \|\cdot\|_{\mathscr{V}})$ is a quasi-normed vector space. Recall that being a quasi-norm amounts to demanding that, for some fixed finite constant $\kappa \geq 2$,

$$\begin{gathered} \|x\|_{\mathscr{V}} = 0 \iff x = 0, \\ \|\lambda x\|_{\mathscr{V}} = |\lambda| \|x\|_{\mathscr{V}} \quad \text{and} \\ \|x + y\|_{\mathscr{V}} \leq \kappa \max\big\{\|x\|_{\mathscr{V}}, \|y\|_{\mathscr{V}}\big\}, \end{gathered} \tag{35}$$

for any vectors $x, y \in \mathscr{V}$ and scalar λ. Given a function $f : X \to \mathscr{V}$ and $r \in (0, +\infty)$, define the oscillation of f at scale r to be

$$\mathrm{osc}_r(f; X, \rho; \mathscr{V}) := \sup\{\|f(x) - f(y)\|_{\mathscr{V}} : x, y \in X, \rho(x, y) \leq r\}. \tag{36}$$

Several important smoothness classes of functions on (X, ρ) are then defined in terms of the growth of the oscillation relative to that of the scale. For example, membership to the class $\mathrm{Lip}(X, \rho; \mathscr{V})$, of $\mathscr{V}$-valued Lipschitz functions on (X, ρ), is characterized by the condition

$$\sup_{r>0} \frac{\mathrm{osc}_r(f; X, \rho; \mathscr{V})}{r} < +\infty \tag{37}$$

and, more generally, membership to the class $\dot{\mathscr{C}}^{\beta}(X,\rho;\mathscr{V})$, of $\mathscr{V}$-valued Hölder functions of order $\beta\in(0,+\infty)$ on (X,ρ), is characterized by the condition

$$\sup_{r>0}\frac{\mathrm{osc}_r(f;X,\rho;\mathscr{V})}{r^{\beta}}<+\infty. \tag{38}$$

Moving on, we wish to extend the scope of the above considerations by considering a more general class of functions. Specifically, let (X,ρ) be a quasi-metric space, $(\mathscr{V},\|\cdot\|_{\mathscr{V}})$ a quasi-normed vector space, and fix a function

$$\omega:(0,+\infty)\longrightarrow(0,+\infty). \tag{39}$$

Also, assume that $E\subseteq X$ is a fixed set of cardinality at least two. Given a function $f:E\to\mathscr{V}$, define its (ω,ρ)-`seminorm` by setting

$$\|f\|_{\dot{\mathscr{C}}^{\omega}(E,\rho;\mathscr{V})}:=\sup_{x,y\in E,x\neq y}\frac{\|f(x)-f(y)\|_{\mathscr{V}}}{\omega(\rho(x,y))}, \tag{40}$$

and introduce the vector space

$$\dot{\mathscr{C}}^{\omega}(E,\rho;\mathscr{V}):=\big\{f:E\to\mathscr{V}:\|f\|_{\dot{\mathscr{C}}^{\omega}(E,\rho;\mathscr{V})}<+\infty\big\}. \tag{41}$$

It is straightforward to check that the quotient space $\dot{\mathscr{C}}^{\omega}(E,\rho;\mathscr{V})/\sim$, where $f\sim g$ provided $f-g$ is constant on E, is a quasi-normed vector space when equipped with the seminorm in (40). Moreover, this quasi-normed vector space is actually a quasi-Banach space (i.e., it is also complete) if $(\mathscr{V},\|\cdot\|_{\mathscr{V}})$ is a quasi-Banach space. It is also clear that replacing the original quasi-norm $\|\cdot\|_{\mathscr{V}}$ by another quasi-norm $\|\cdot\|'_{\mathscr{V}}$ on $\mathscr{V}$ which is equivalent with it (in the sense that there exists $C\in[1,+\infty)$ such that $C^{-1}\|\cdot\|_{\mathscr{V}}\leq\|\cdot\|'_{\mathscr{V}}\leq C\|\cdot\|_{\mathscr{V}}$ on $\mathscr{V}$) changes (40) into a quasi-norm equivalent with it.

For further reference, let us also note here that

$$\omega\text{ non-decreasing}\Longrightarrow\|f\|_{\dot{\mathscr{C}}^{\omega}(E,\rho;\mathscr{V})}=\sup_{r>0}\frac{\mathrm{osc}_r(f;E,\rho;\mathscr{V})}{\omega(r)}, \tag{42}$$

(interpreting (E,ρ) as a quasi-metric space in its own right), and observe that if $\mathscr{C}^0(E;\mathscr{V})$ denotes the space of $\mathscr{V}$-valued continuous functions on E (relative to the topology induced by τ_ρ on E) then

$$\lim_{t\to0^+}\omega(t)=0\Longrightarrow\dot{\mathscr{C}}^{\omega}(E,\rho;\mathscr{V})\subseteq\mathscr{C}^0(E;\mathscr{V}). \tag{43}$$

Convention 1. *Throughout the paper, we agree to drop the dependence on the quasi-normed vector space $\mathscr{V}$ in the special case when $\mathscr{V}=\mathbb{R}$. In particular, we abbreviate*

$$\dot{\mathscr{C}}^{\omega}(E,\rho):=\dot{\mathscr{C}}^{\omega}(E,\rho;\mathbb{R}),\quad\mathscr{C}^0(E):=\mathscr{C}^0(E;\mathbb{R}),\ \textit{et cetera}. \tag{44}$$

The proposition below deals with the issue of the membership to $\dot{\mathscr{C}}^{\omega}(E,\rho)$ of the pointwise supremum of a family of functions from this space.

Proposition 2. *Let (X,ρ) be a quasi-metric space, fix a function ω as in (39), and suppose that $E\subseteq X$ is a set of cardinality at least two. Given a family $\{f_i\}_{i\in I}$ of real-valued functions defined on E with the property that*

$$M:=\sup_{i\in I}\|f_i\|_{\dot{\mathscr{C}}^{\omega}(E,\rho)}<+\infty, \tag{45}$$

consider

$$f^*(x):=\sup_{i\in I}f_i(x),\qquad\textit{for every }x\in E. \tag{46}$$

Then either $f^*(x) = +\infty$ for every $x \in E$, or $f^* : E \to \mathbb{R}$ is a well-defined function satisfying $\|f^*\|_{\dot{\mathscr{C}}^\omega(E,\rho)} \leq M$.

Proof. If f^* is not identically $+\infty$ on E then there exists $x_0 \in E$ such that $\sup_{i\in I} f_i(x_0) < +\infty$. On the other hand, condition (45) entails that, for each $i \in I$,

$$f_i(x) \leq f_i(y) + M\omega(\rho(x,y)), \qquad \forall x, y \in E \text{ with } x \neq y. \tag{47}$$

Using (47) with $y := x_0$ then gives

$$\sup_{i\in I} f_i(x) \leq \sup_{i\in I} f_i(x_0) + M\omega(\rho(x,x_0)) < +\infty, \qquad \text{for every } x \in E. \tag{48}$$

Thus, the function $f^* : E \to \mathbb{R}$ given by $f^*(x) := \sup_{i\in I} f_i(x)$ for each $x \in E$ is well-defined. Moreover, (47) readily gives that $f^*(x) \leq f^*(y) + M\omega(\rho(x,y))$ for all $x, y \in E$ with $x \neq y$ hence, ultimately, $\|f^*\|_{\dot{\mathscr{C}}^\omega(E,\rho)} \leq M$. This finishes the proof of the proposition. □

As a corollary of Proposition 2, we note that if (X, ρ) is a quasi-metric space, $E \subseteq X$ is a set of cardinality at least two, and if ω is as in (39), then for any finite family of functions $f_i \in \dot{\mathscr{C}}^\omega(E,\rho)$, $1 \leq i \leq N$, $N \in \mathbb{N}$, it follows that

$$\max_{1\leq i\leq N} f_i \in \dot{\mathscr{C}}^\omega(E,\rho), \qquad \min_{1\leq i\leq N} f_i \in \dot{\mathscr{C}}^\omega(E,\rho), \tag{49}$$

and

$$\max\Big\{\big\|\max_{1\leq i\leq N} f_i\big\|_{\dot{\mathscr{C}}^\omega(E,\rho)}, \big\|\min_{1\leq i\leq N} f_i\big\|_{\dot{\mathscr{C}}^\omega(E,\rho)}\Big\} \leq \max_{1\leq i\leq N} \|f_i\|_{\dot{\mathscr{C}}^\omega(E,\rho)}. \tag{50}$$

In closing, we alert the reader to the fact that, given a set E, we shall use $\mathbf{1}_E$ to denote the characteristic function of E, and $\#E$ to denote the cardinality of E.

3. **The extension problem on general quasi-metric spaces.** This section is devoted to proving Theorem 3.3, which states (using notation introduced in §2) that any function in $\dot{\mathscr{C}}^\omega(E,\rho)$ may be extended to a function in $\dot{\mathscr{C}}^\omega(X,\rho)$, under suitable background assumptions on ω. The extension result presented in Theorem 3.3, which generalizes classical work by McShane in the context of Lipschitz functions on metric spaces, should be compared with Theorem 7.1, stated in §7. Specifically, while the extension procedure employed in the proof of Theorem 3.3 is nonlinear, this result is valid in quasi-metric spaces of general nature. By way of contrast, while the extension scheme devised in the proof of Theorem 7.1 is linear, this presupposes that the quasi-metric space in question is geometrically doubling (as described in Definition 6.1).

Turning to specifics, we debut by making the following definition.

Definition 3.1. Given $\beta \in (0, +\infty)$, call a function $\omega : [0, +\infty) \to [0, +\infty)$ a β-`modulation` provided

$$\omega \text{ is non-decreasing on } [0,+\infty), \quad \omega(t) > 0 \text{ for } t > 0, \text{ and} \tag{51}$$

$$\omega(r) \leq \inf\big\{\omega(s) + \omega(t) : s, t \geq 0, s^\beta + t^\beta = r^\beta\big\} \text{ for all } r \geq 0. \tag{52}$$

Furthermore, under the additional assumption that

$$\omega \text{ continuously vanishes at the origin,} \tag{53}$$

we shall refer to the β-modulation ω as being a β-`modulus of continuity`.

Occasionally we will call the property described in (52) the β-*subadditivity condition* for ω. A prototypical example of a β-modulus of continuity is

$$\omega_{c,\beta} : \big[0,+\infty) \longrightarrow [0,+\infty), \quad \omega_{c,\beta}(t) := ct^{\beta} \text{ for each } t \in [0,+\infty), \tag{54}$$

where $c, \beta \in (0,+\infty)$ are given. Other useful properties of the class of modulations we are considering are collected in the lemma below. To state it, set $\mathbb{N}_0 := \{0,1,2,...\}$ and, for each $a \in \mathbb{R}$, define

$$\langle a \rangle := \begin{cases} \inf\{n \in \mathbb{N}_0 : a \leq n\} & \text{if } a \geq 0, \\ 0 & \text{if } a < 0. \end{cases} \tag{55}$$

Lemma 3.2. *Let $\beta \in (0,+\infty)$ and assume that the function $\omega : [0,+\infty) \to [0,+\infty)$ is a β-modulation. Then the following properties hold.*

(i) The function ω is a β'-modulation for any $\beta' \in (\beta,+\infty)$.
(ii) For every $\gamma \in (0,+\infty)$ the function $[0,+\infty) \ni r \mapsto \omega(r^{\gamma}) \in [0,+\infty)$ is a $(\beta\gamma)$-modulation. In particular, the function $\widetilde{\omega} : [0,+\infty) \to [0,+\infty)$ given by $\widetilde{\omega}(r) := \omega(r^{1/\beta})$ for every $r \geq 0$, is subadditive.
(iii) If the given function ω continuously vanishes at the origin (i.e., if ω is a β-modulus of continuity), then ω is continuous on $[0,+\infty)$.
(iv) The function ω satisfies the following slow-growth property:

$$\omega(ct) \leq 2^{\langle \beta \log_2 c \rangle} \omega(t) \quad \textit{for each } t \in [0,+\infty) \textit{ and each } c \in (0,+\infty). \tag{56}$$

Proof. To prove (i) fix $r \in [0,+\infty)$ and let $s,t \in [0,+\infty)$ be such that $r^{\beta'} = s^{\beta'} + t^{\beta'}$. Then,

$$r^{\beta} = \big(r^{\beta'}\big)^{\beta/\beta'} = \big(s^{\beta'} + t^{\beta'}\big)^{\beta/\beta'} \leq \big(s^{\beta'}\big)^{\beta/\beta'} + \big(t^{\beta'}\big)^{\beta/\beta'} = s^{\beta} + t^{\beta}, \tag{57}$$

where the inequality above follows from the fact that $\beta \leq \beta'$. Then (i) is an immediate consequence of (57), the monotonicity of ω, and the observation that the condition (52) implies

$$\omega\big((s^{\beta} + t^{\beta})^{1/\beta}\big) \leq \omega(s) + \omega(t) \quad \text{for all } s,t \geq 0. \tag{58}$$

Next, (ii) follows directly from definitions. Turning our attention to (iii) we first recall a basic result regarding subadditive functions (cf., e.g., [13] and [26]) according to which if $f : [0,+\infty) \to \mathbb{R}$ is subadditive and continuously vanishing at the origin then for each $t > 0$ the one-sided limits $f(t^{\pm}) := \lim_{s \to t^{\pm}} f(s)$ exist and $f(t^{+}) \leq f(t) \leq f(t^{-})$. Using (ii) and applying the above result to the subadditive function $f := \widetilde{\omega}$ from (ii), we may conclude (since if ω continuously vanishes at the origin then so does $\widetilde{\omega}$) that $\widetilde{\omega}(r^{+}) \leq \widetilde{\omega}(r) \leq \widetilde{\omega}(r^{-})$ for each $r > 0$. In turn, since $\widetilde{\omega}$ is non-decreasing, this shows that the function $\widetilde{\omega}$ is continuous on $[0,+\infty)$. Hence, ω itself is continuous on $[0,+\infty)$, completing the proof of (iii).

We are left with proving (iv). Since when $0 < c \leq 1$ the estimate (56) is a direct consequence of (51) and (55), there remains to consider the situation when $c > 1$. Assume that this is the case and note that, when specialized to the case when $s = t$, (58) yields

$$\omega\big(2^{1/\beta} t\big) \leq 2\omega(t) \quad \text{for all } t \geq 0. \tag{59}$$

Furthermore, iterating the inequality in (59) yields

$$\omega\big(2^{n/\beta} t\big) \leq 2^{n}\omega(t) \quad \text{for all } t \geq 0 \text{ and all } n \in \mathbb{N}_0. \tag{60}$$

Finally, if we set $n := \langle \beta \log_2 c \rangle \in \mathbb{N}_0$ then $c \leq 2^{n/\beta}$ which, in conjunction with (51) and (60), gives

$$\omega(ct) \leq \omega(2^{n/\beta} t) \leq 2^n \omega(t) \quad \text{ for all } t \geq 0, \tag{61}$$

finishing the proof of (56). □

Remark 2. The slow-growth property (56) of a β-modulation ω has the following significant consequence. If X is a given set, E is a subset of X of cardinality at least two, and if $\rho, \rho' \in \mathfrak{Q}(X)$ are equivalent quasi-distances on X (cf. (14)), then $\dot{\mathscr{C}}^\omega(E,\rho)$ and $\dot{\mathscr{C}}^\omega(E,\rho')$ coincide as sets, while $\|\cdot\|_{\dot{\mathscr{C}}^\omega(E,\rho)}$ and $\|\cdot\|_{\dot{\mathscr{C}}^\omega(E,\rho')}$ are equivalent semi-norms.

We are now ready to present the generalization of the work by E.J. McShane [22] and M.D. Kirszbraun [17] mentioned at the beginning of this section.

Theorem 3.3. *Let (X,ρ) be a quasi-metric space and suppose that $E \subseteq X$ has cardinality at least two. Also, fix a finite number $\beta \in (0, (\log_2 C_\rho)^{-1}]$, where C_ρ is as in (15), and consider a β-modulation ω with the property that $\omega(0) = 0$. Then any function belonging to $\dot{\mathscr{C}}^\omega(E,\rho)$ may be extended to the entire space X with preservation of smoothness, while retaining control of the associated semi-norm. More specifically, with $\widetilde{C}_\rho$ as in (16),*

$$\begin{array}{l} \textit{for every } f \in \dot{\mathscr{C}}^\omega(E,\rho) \textit{ there exists } F \in \dot{\mathscr{C}}^\omega(X,\rho) \textit{ for which} \\ f = F|_E \textit{ and } \|F\|_{\dot{\mathscr{C}}^\omega(X,\rho)} \leq 2^{\langle 2\beta \log_2 C_\rho \rangle + \langle \beta \log_2 \widetilde{C}_\rho \rangle} \|f\|_{\dot{\mathscr{C}}^\omega(E,\rho)}. \end{array} \tag{62}$$

As a corollary,

$$\dot{\mathscr{C}}^\omega(E,\rho) = \big\{ F|_E : F \in \dot{\mathscr{C}}^\omega(X,\rho) \big\}. \tag{63}$$

Proof. To get started, we note that Theorem 2.1 ensures the existence of some $\rho_\# \in \mathfrak{Q}(X)$ with the property that $(\rho_\#)^\beta$ is a distance on X and such that $\rho_\# \approx \rho$. More specifically (cf. (27), (33), (24)),

$$(C_\rho)^{-2} \rho \leq \rho_\# \leq \widetilde{C}_\rho \rho \qquad \text{on } X \times X. \tag{64}$$

Given an arbitrary function $f \in \dot{\mathscr{C}}^\omega(E,\rho)$, consider the constant

$$K := 2^{\langle 2\beta \log_2 C_\rho \rangle} \|f\|_{\dot{\mathscr{C}}^\omega(E,\rho)} \in [0, +\infty) \tag{65}$$

and, for each $z \in E$, define the function

$$f_z : X \longrightarrow \mathbb{R}, \qquad f_z(x) := f(z) - K\omega(\rho_\#(x,z)) \quad \forall x \in X. \tag{66}$$

The choice of K in (65) ensures that

$$f_z \leq f \text{ pointwise on } E, \text{ for each fixed } z \in E. \tag{67}$$

Indeed, for each $x, z \in E$ we may write, based on (64), the monotonicity of ω, the slow-growth property (56), and the definition of K,

$$\begin{aligned} |f(x) - f(z)| &\leq \|f\|_{\dot{\mathscr{C}}^\omega(E,\rho)} \omega(\rho(x,z)) \leq \|f\|_{\dot{\mathscr{C}}^\omega(E,\rho)} \omega\big(C_\rho^2 \rho_\#(x,z)\big) \\ &\leq \|f\|_{\dot{\mathscr{C}}^\omega(E,\rho)} 2^{\langle \beta \log_2 C_\rho^2 \rangle} \omega\big(\rho_\#(x,z)\big) \\ &= K\omega\big(\rho_\#(x,z)\big). \end{aligned} \tag{68}$$

With this in hand, for each $x, z \in E$ we then obtain

$$f_z(x) - f(x) = f(z) - f(x) - K\omega(\rho_\#(x,z)) \leq 0, \tag{69}$$

from which (67) follows. Next we claim that the function

$$F : X \longrightarrow \mathbb{R}, \qquad F(x) := \sup_{z\in E} f_z(x) \quad \forall x \in X, \tag{70}$$

is well-defined and satisfies the properties listed in the second line of (62). With this goal in mind, we first note that for each $x, y, z \in X$ we have

$$\omega\big(\rho_\#(x,y)\big) \leq \omega\big((\rho_\#(x,z)^\beta + \rho_\#(z,y)^\beta)^{1/\beta}\big) \leq \omega\big(\rho_\#(x,z)\big) + \omega\big(\rho_\#(z,y)\big), \tag{71}$$

since $(\rho_\#)^\beta$ is a distance on X, and since ω is non-decreasing and β-subadditive. In turn, given that $\rho_\#$ is symmetric, (71) entails

$$\big|\omega\big(\rho_\#(x,z)\big) - \omega\big(\rho_\#(y,z)\big)\big| \leq \omega\big(\rho_\#(x,y)\big), \qquad \forall x, y, z \in X. \tag{72}$$

Thus, for each $z \in E$ and $x, y \in X$

$$\begin{aligned} |f_z(x) - f_z(y)| \;&=\; K\big|\omega\big(\rho_\#(x,z)\big) - \omega\big(\rho_\#(y,z)\big)\big| \\ &\leq K\omega(\rho_\#(x,y)) \leq K\omega\big(\widetilde{C}_\rho \rho(x,y)\big) \\ &\leq K 2^{\langle \beta \log_2 \widetilde{C}_\rho\rangle}\omega(\rho(x,y)), \end{aligned} \tag{73}$$

where the first inequality in (73) follows from (72), the second is a consequence of (51) and (64), while the third follows from (51) and (56). Hence, for each $z \in E$, we have $f_z \in \dot{\mathscr{C}}^\omega(X,\rho)$ and

$$\sup_{z\in E} \|f_z\|_{\dot{\mathscr{C}}^\omega(X,\rho)} \leq K 2^{\langle \beta \log_2 \widetilde{C}_\rho\rangle} < +\infty. \tag{74}$$

In turn, from this, (70), (67) and Proposition 2 we obtain that F is well-defined, $F \leq f$ on E and $\|F\|_{\dot{\mathscr{C}}^\omega(X,\rho)} \leq K 2^{\langle \beta \log_2 \widetilde{C}_\rho\rangle} = 2^{\langle 2\beta \log_2 C_\rho\rangle + \langle \beta \log_2 \widetilde{C}_\rho\rangle}\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho)}$. Moreover, since $F(z) \geq f_z(z) = f(z)$ for each $z \in E$ (here we have used the fact that ω vanishes at the origin), we also obtain that $F \geq f$ on E. Hence, ultimately, $F = f$ on E which completes the proof of (62). Finally, (63) is a simple consequence of (62) and the observation that under pointwise restriction one retains control of the (ω, β)-seminorm. □

In turn, the extension result presented in Theorem 3.3 is going to play a key role in establishing a quantitative version of the classical Urysohn's lemma in the next section.

4. **A quantitative Urysohn lemma.** The classical Urysohn's lemma (cf., e.g., [24, Theorem 33.1, p. 207]) is a basic result in topology, asserting that if (X, τ) is a locally compact, Hausdorff topological space, F_0, F_1 two nonempty, disjoint subsets of X, such that F_0 is compact and F_1 is closed, then there exists $\psi \in \mathscr{C}^0(X)$ with the property that $0 \leq \psi \leq 1$ on X, $\psi \equiv 0$ on F_0, and $\psi \equiv 1$ on F_1. Our goal in this section is to present a quantitative version of this result, as described in the theorem below.

Theorem 4.1. *Let (X, ρ) be a quasi-metric space and let $C_\rho, \widetilde{C}_\rho \in [1, +\infty)$ be as in (15)-(16). Fix a finite number $\beta \in (0, (\log_2 C_\rho)^{-1}]$ and consider a β-modulation ω which satisfies $\omega(0) = 0$. Suppose that $F_0, F_1 \subseteq X$ are two nonempty sets with the property that $\mathrm{dist}_\rho(F_0, F_1) > 0$. Then, there exists $\psi \in \dot{\mathscr{C}}^\omega(X,\rho)$ such that*

$$0 \leq \psi \leq 1 \text{ on } X, \quad \psi \equiv 0 \text{ on } F_0, \quad \psi \equiv 1 \text{ on } F_1, \tag{75}$$

and for which

$$\|\psi\|_{\dot{\mathscr{C}}^\omega(X,\rho)} \leq 2^{\langle 2\beta \log_2 C_\rho\rangle + 2\langle \beta \log_2 \widetilde{C}_\rho\rangle}\big[\omega\big(\mathrm{dist}_\rho(F_0, F_1)\big)\big]^{-1}. \tag{76}$$

Proof. Consider the function $\varphi : F_0 \cup F_1 \to \mathbb{R}$ given by

$$\varphi(x) := \begin{cases} 0 & \text{if } x \in F_0, \\ 1 & \text{if } x \in F_1, \end{cases} \qquad x \in F_0 \cup F_1. \tag{77}$$

Notice that if either $x, y \in F_0$, or $x, y \in F_1$, we have

$$|\varphi(x) - \varphi(y)| = 0 \leq 2^{\langle \beta \log_2 \widetilde{C}_\rho \rangle} \big[\omega(\mathrm{dist}_\rho(F_0, F_1))\big]^{-1} \omega(\rho(x, y)), \tag{78}$$

given that $\mathrm{dist}_\rho(F_0, F_1) > 0$. Also, if the point $x \in F_1$ and the point $y \in F_0$ then $0 < \mathrm{dist}_\rho(F_0, F_1) \leq \rho(y, x) \leq \widetilde{C}_\rho \rho(x, y)$. Keeping in mind that ω is non-decreasing this forces

$$\omega(\mathrm{dist}_\rho(F_0, F_1)) \leq \omega\big(\widetilde{C}_\rho \rho(x, y)\big) \leq 2^{\langle \beta \log_2 \widetilde{C}_\rho \rangle} \omega(\rho(x, y)), \tag{79}$$

by (56). Consequently,

$$|\varphi(x) - \varphi(y)| = 1 \leq 2^{\langle \beta \log_2 \widetilde{C}_\rho \rangle} \big[\omega(\mathrm{dist}_\rho(F_0, F_1))\big]^{-1} \omega(\rho(x, y)). \tag{80}$$

In fact, a similar (and simpler) reasoning shows that (80) continues to be true in the case when $x \in F_0$ and $y \in F_1$, as well. All together, these imply

$$\varphi \in \dot{\mathscr{C}}^\omega(F_0 \cup F_1, \rho) \ \text{ and } \ \|\varphi\|_{\dot{\mathscr{C}}^\omega(F_0 \cup F_1, \rho)} \leq 2^{\langle \beta \log_2 \widetilde{C}_\rho \rangle} \big[\omega(\mathrm{dist}_\rho(F_0, F_1))\big]^{-1}. \tag{81}$$

With this in hand, Theorem 3.3 then ensures the existence of $\tilde{\varphi} \in \dot{\mathscr{C}}^\omega(X, \rho)$ which extends the function φ and which has the property that

$$\|\tilde{\varphi}\|_{\dot{\mathscr{C}}^\omega(X, \rho)} \leq 2^{\langle 2\beta \log_2 C_\rho \rangle + 2\langle \beta \log_2 \widetilde{C}_\rho \rangle} \big[\omega(\mathrm{dist}_\rho(F_0, F_1))\big]^{-1}. \tag{82}$$

At this stage, consider $\psi : X \to \mathbb{R}$ given by

$$\psi := \min\big\{\max\{\tilde{\varphi}, 0\}, 1\big\}. \tag{83}$$

By design, the function ψ satisfies (75). Moreover, (49)-(50) yield $\psi \in \dot{\mathscr{C}}^\omega(X, \rho)$ as well as the estimate $\|\psi\|_{\dot{\mathscr{C}}^\omega(X, \rho)} \leq \|\tilde{\varphi}\|_{\dot{\mathscr{C}}^\omega(X, \rho)}$. This and (82) then prove (76), completing the proof of the theorem. □

5. **Whitney-like partitions of unity via $\dot{\mathscr{C}}^\omega$ functions.** An important tool in Harmonic Analysis is the Whitney decomposition of an open, nonempty, proper subset $\mathcal{O}$ of a quasi-metric space (X, ρ) into ρ-balls whose location is related to their distance to the complement of $\mathcal{O}$ in X (in a sense to be made precise later). Frequently, given such a Whitney decomposition, it is useful to have a partition of unity subordinate to it, which is quantitative in the sense that the size of the functions involved is controlled in terms of the size of their respective supports. Details in the standard setting of the Euclidean space $\mathbb{R}^n$ may be found in, e.g., [29, p. 170].

More recently, such quantitative Whitney partitions of unity have been constructed on general metric spaces (see [18, Lemma 2.4, p.339], [11]), and on quasi-metric spaces, as in [21, Lemma 2.16, p. 278]. Here we wish to improve upon the latter result both by allowing more general set-theoretic and functional-analytic frameworks, as described in the theorem below.

Theorem 5.1. *Let (X, ρ) be a quasi-metric space and let the finite constants $C_\rho, \widetilde{C}_\rho \in [1, +\infty)$ be as in (15)-(16). Also, fix a finite number $\beta \in (0, (\log_2 C_\rho)^{-1}]$ and consider a β-modulation ω with the property that $\omega(0) = 0$. In this setting, assume that $\{E_j\}_{j \in I}$, $\{\widetilde{E}_j\}_{j \in I}$ and $\{\widehat{E}_j\}_{j \in I}$ are three families of nonempty proper subsets of X satisfying the following properties:*

(a) for each $j \in I$ one has $E_j \subseteq \widetilde{E}_j \subseteq \widehat{E}_j$, $r_j := \mathrm{dist}_\rho(E_j, X \setminus \widetilde{E}_j) > 0$ and

$$\mathrm{dist}_\rho(\widetilde{E}_j, X \setminus \widehat{E}_j) \approx r_j \quad \text{uniformly for } j \in I; \tag{84}$$

(b) one has $r_i \approx r_j$ uniformly for $i, j \in I$ such that $\widehat{E}_i \cap \widehat{E}_j \neq \emptyset$;

(c) there exists $N \in \mathbb{N}$ such that $\sum_{j \in I} \mathbf{1}_{\widehat{E}_j} \leq N$;

(d) one has $\bigcup_{j \in I} E_j = \bigcup_{j \in I} \widehat{E}_j$.

Then there exists a finite constant $C \geq 1$, depending only on $C_\rho, \widetilde{C}_\rho \in [1, +\infty)$, β, N, and the proportionality constants in (a) and (b) above, along with a family of real-valued functions $\{\varphi_j\}_{j \in I}$ defined on X such that the following conditions are valid:

(1) for each $j \in I$ one has

$$\varphi_j \in \dot{\mathscr{C}}^\omega(X, \rho) \quad \text{and} \quad \|\varphi_j\|_{\dot{\mathscr{C}}^\omega(X,\rho)} \leq C\omega(r_j)^{-1}; \tag{85}$$

(2) for every $j \in I$ one has

$$0 \leq \varphi_j \leq 1 \text{ on } X, \quad \varphi_j \equiv 0 \text{ on } X \setminus \widetilde{E}_j, \quad \text{and} \quad \varphi_j \geq 1/C \text{ on } E_j; \tag{86}$$

(3) one has $\sum_{j \in I} \varphi_j = \mathbf{1}_{\bigcup_{j \in I} E_j} = \mathbf{1}_{\bigcup_{j \in I} \widetilde{E}_j} = \mathbf{1}_{\bigcup_{j \in I} \widehat{E}_j}$.

Proof. Based on Theorem 4.1 and property *(a)*, for each $j \in I$ there exists a function $\psi_j \in \dot{\mathscr{C}}^\omega(X, \rho)$ such that

$$(i)\ \psi_j \equiv 1 \text{ on } E_j, \quad (ii)\ \psi_j \equiv 0 \text{ on } X \setminus \widetilde{E}_j, \quad (iii)\ 0 \leq \psi_j \leq 1 \text{ on } X, \tag{87}$$

and

$$\|\psi_j\|_{\dot{\mathscr{C}}^\omega(X,\rho)} \leq 2^{\langle 2\beta \log_2 C_\rho \rangle + 2\langle \beta \log_2 \widetilde{C}_\rho \rangle} \omega(r_j)^{-1}. \tag{88}$$

Consider next the function

$$\Psi : \bigcup_{j \in I} E_j \longrightarrow \mathbb{R}, \quad \Psi := \sum_{j \in I} \psi_j \quad \text{on} \quad \bigcup_{j \in I} E_j, \tag{89}$$

and note that Ψ is well-defined and satisfies

$$1 \leq \Psi \leq N \quad \text{on} \quad \bigcup_{j \in I} E_j. \tag{90}$$

Indeed, the fact that Ψ is well-defined follows from *(c)* and (iii) in (87), the first inequality in (90) is due to (i) and (iii) in (87) and the second inequality above is a consequence of (iii) in (87), the fact that $E_j \subseteq \widehat{E}_j$ for each $j \in I$, and statement *(c)* in the hypotheses. Going further, for each $j \in I$ introduce the function

$$\varphi_j : X \longrightarrow \mathbb{R}, \quad \varphi_j := \begin{cases} \psi_j/\Psi & \text{on } \bigcup_{i \in I} E_i, \\ 0 & \text{on } X \setminus \big(\bigcup_{i \in I} E_i\big). \end{cases} \tag{91}$$

By the above discussion, for each $j \in I$ the function φ_j is well-defined and, thanks to (91), the first inequality in (90) and $(ii) - (iii)$ in (87), satisfies

$$0 \leq \varphi_j \leq 1 \ \text{ on } \ X, \ \text{and} \ \ \varphi_j \equiv 0 \ \text{ on } \ X \setminus \widetilde{E}_j. \tag{92}$$

This proves the first two assertions in *(2)* in the conclusion of the theorem. Also, employing (91), (i) in (87), and the second inequality in (90), we may conclude that

$$\varphi_j = \psi_j/\Psi = 1/\Psi \geq 1/N \quad \text{on} \quad E_j. \tag{93}$$

This finishes the proof of *(2)* provided one chooses $C \geq N$. Going further, by *(c)* and the second property in (92), the sum $\sum_{j\in I}\varphi_j$ is meaningfully defined in $\mathbb{R}$. In fact, from (91) and (89), this sum is identically equal to one on $\bigcup_{j\in I} E_j$. Using this analysis and *(d)* finishes the proof of conclusion *(3)* from the statement of the theorem.

There remains to prove *(1)*. To this end, as a preliminary step we will show that for each $j \in I$, there holds

$$|\psi_j(x)-\psi_j(y)| \leq 2^{\langle 2\beta\log_2 C_\rho\rangle+2\langle\beta\log_2\widetilde{C}_\rho\rangle}\omega(r_j)^{-1}\omega(\rho(x,y)) \times\big[\mathbf{1}_{\widehat{E}_j}(x)+\mathbf{1}_{\widehat{E}_j}(y)\big], \qquad \forall x,y\in X. \tag{94}$$

In order to prove (94), fix $j \in I$ and, based on the fact that $\psi_j \in \dot{\mathscr{C}}^\omega(X,\rho)$ and (88), estimate for all $x, y \in X$,

$$\begin{aligned}|\psi_j(x)-\psi_j(y)| &\leq \|\psi_j\|_{\dot{\mathscr{C}}^\omega(X,\rho)}\omega(\rho(x,y))\\ &\leq 2^{\langle 2\beta\log_2 C_\rho\rangle+2\langle\beta\log_2\widetilde{C}_\rho\rangle}\omega(r_j)^{-1}\omega(\rho(x,y)).\end{aligned}\tag{95}$$

By construction, $\psi_j \equiv 0$ on $X\setminus\widetilde{E}_j$ so that if $x,y\in X\setminus\widetilde{E}_j$ then (94) is obviously true. In the case when either $x\in\widetilde{E}_j$ or $y\in\widetilde{E}_j$, using the fact that $\widetilde{E}_j\subseteq\widehat{E}_j$ we may write

$$\mathbf{1}_{\widehat{E}_j}(x)+\mathbf{1}_{\widehat{E}_j}(y)\geq 1, \tag{96}$$

and thus (94) follows from (96) and (95), in this case. This finishes the justification of (94).

Having disposed of this, we now focus on proving *(1)*, i.e., show that for each fixed $j \in I$

$$|\varphi_j(x)-\varphi_j(y)|\leq C\omega(r_j)^{-1}\omega(\rho(x,y)), \quad \forall x,y\in X, \tag{97}$$

for some finite constant $C > 0$, depending only on $C_\rho, \widetilde{C}_\rho \in [1,+\infty)$, β, N, and the proportionality constants in conditions *(a)* and *(b)*. Fix $j \in I$ and note that (97) is obviously true whenever $x,y\in X\setminus\big(\bigcup_{i\in I}E_i\big)$ as the left-hand side in (97) vanishes in this case (cf. (91)). Consider next the case when $x,y\in\bigcup_{i\in I}E_i$ in which scenario we compute

$$\begin{aligned}|\varphi_j(x)-\varphi_j(y)| &= \left|\frac{\psi_j(x)}{\Psi(x)}-\frac{\psi_j(y)}{\Psi(y)}\right| = \left|\frac{\psi_j(x)\Psi(y)-\psi_j(y)\Psi(x)}{\Psi(x)\Psi(y)}\right|\\ &\leq |\psi_j(x)\Psi(y)-\psi_j(y)\Psi(x)|\\ &\leq |\psi_j(x)-\psi_j(y)|\Psi(y)+|\Psi(x)-\Psi(y)|\psi_j(y)\\ &\leq N|\psi_j(x)-\psi_j(y)|+|\Psi(x)-\Psi(y)|\mathbf{1}_{\widetilde{E}_j}(y) =: I_1+I_2.\end{aligned}\tag{98}$$

The first inequality above follows from the first inequality in (90), the second estimate is a consequence of the triangle inequality, and the third one follows from (90) and (ii)-(iii) in (87). Moving on, as a direct consequence of (95) we obtain

$$I_1\leq 2^{\langle 2\beta\log_2 C_\rho\rangle+2\langle\beta\log_2\widetilde{C}_\rho\rangle}N\omega(r_j)^{-1}\omega(\rho(x,y)), \qquad \forall x,y\in X. \tag{99}$$

As for I_2 we make the claim that there exists a finite constant $C > 0$, depending only on $C_\rho, \widetilde{C}_\rho \in [1,+\infty)$, β, N and the proportionality constants in conditions *(a)* and *(b)* from the hypotheses, such that

$$I_2\leq C\omega(r_j)^{-1}\omega(\rho(x,y)), \qquad \forall x,y\in\bigcup_{i\in I}E_i. \tag{100}$$

To justify this claim, observe that if $y \in \big(\bigcup_{i\in I} E_i\big) \setminus \widetilde{E}_j$ then $I_2 = 0$, so estimate (100) is trivially true. Consider next the case when $y \in \big(\bigcup_{i\in I} E_i\big) \cap \widetilde{E}_j$ and denote by $c > 0$ the lower proportionality constant in (84). If $\rho(x,y) \geq cr_j/\widetilde{C}_\rho$ then, on the one hand, (51) and the slow-growth condition for ω described in (56) imply the existence of a finite constant $C = C(\rho,\beta,c) > 0$ such that $\omega(r_j) \leq C\omega(\rho(x,y))$, while on the other hand $I_2 \leq 2N$ by the second inequality in (90). Hence (100) holds in this case as well, if C is sufficiently large. Note that $\rho(x,y) < cr_j/\widetilde{C}_\rho$ forces $\rho(y,x) < cr_j$. Therefore, suppose next that

$$x \in \bigcup_{i\in I} E_i \quad \text{and} \quad y \in \big(\bigcup_{i\in I} E_i\big) \cap \widetilde{E}_j \quad \text{are such that} \quad \rho(y,x) < cr_j. \tag{101}$$

Given that $y \in \widetilde{E}_j$ and since by (84) and (101) we have

$$\mathrm{dist}_\rho(\widetilde{E}_j, X \setminus \widehat{E}_j) \geq cr_j > \rho(y,x), \tag{102}$$

which further entails $x \in \widehat{E}_j$. Based on this, the triangle inequality and (94), it follows that

$$\begin{aligned} I_2 &= |\Psi(x) - \Psi(y)|\mathbf{1}_{\widehat{E}_j}(x)\mathbf{1}_{\widehat{E}_j}(y) \leq \sum_{i\in I} |\psi_i(x) - \psi_i(y)|\mathbf{1}_{\widehat{E}_j}(x)\mathbf{1}_{\widehat{E}_j}(y) \\ &\leq 2^{\langle 2\beta \log_2 C_\rho\rangle + 2\langle \beta \log_2 \widetilde{C}_\rho\rangle}\omega(\rho(x,y)) \sum_{i\in I} \omega(r_i)^{-1}\big[\mathbf{1}_{\widehat{E}_i}(x) + \mathbf{1}_{\widehat{E}_i}(y)\big]\mathbf{1}_{\widehat{E}_j}(x)\mathbf{1}_{\widehat{E}_j}(y) \\ &\leq 2^{\langle 2\beta \log_2 C_\rho\rangle + 2\langle \beta \log_2 \widetilde{C}_\rho\rangle}\omega(\rho(x,y))\big\{I_2^{'} + I_2^{''}\big\}, \quad \text{if } x,y \text{ are as in (101)}, \end{aligned} \tag{103}$$

where

$$I_2^{'} := \sum_{i\in I} \omega(r_i)^{-1}\mathbf{1}_{\widehat{E}_i}(x)\mathbf{1}_{\widehat{E}_j}(x) \quad \text{and} \quad I_2^{''} := \sum_{i\in I} \omega(r_i)^{-1}\mathbf{1}_{\widehat{E}_i}(y)\mathbf{1}_{\widehat{E}_j}(y). \tag{104}$$

For each non-zero term in $I_2^{'}$ we necessarily have $x \in \widehat{E}_i \cap \widehat{E}_j$ hence $\widehat{E}_i \cap \widehat{E}_j \neq \emptyset$, which further forces $r_i \approx r_j$, by condition *(b)* in the hypotheses. Thus, using this, property *(c)* from the hypotheses, and (56),

$$I_2^{'} \leq C\omega(r_j)^{-1} \sum_{i\in I} \mathbf{1}_{\widehat{E}_i}(x) \leq CN\omega(r_j)^{-1}, \tag{105}$$

where $C > 0$ is a finite constant which depends only on β and the proportionality constant in *(b)*. Similarly, $I_2^{''} \leq CN\omega(r_j)^{-1}$ for some finite constant $C > 0$ depending only on β and the proportionality constant in *(b)*. Granted the discussion in the paragraph above (101), it follows from this and (103) that (100) holds as stated.

In summary, this analysis shows that the estimate in (97) holds whenever either $x, y \in X \setminus \big(\bigcup_{i\in I} E_i\big)$, or $x, y \in \bigcup_{i\in I} E_i$. Therefore, in order to finish the proof of (97) it remains to establish this inequality in the case when

$$x \in \bigcup_{i\in I} E_i \quad \text{and} \quad y \in X \setminus \big(\bigcup_{i\in I} E_i\big), \tag{106}$$

or vice-versa. Concretely, assume that (106) holds (the other case is treated similarly). Then (97) is clear when $x \notin \widetilde{E}_j$ since in such a scenario $\varphi_j(x) = \varphi_j(y) = 0$ by the second property in (92) and the second condition in (106). Thus matters have been reduced to considering the case when

$$x \in \big(\bigcup_{i\in I} E_i\big) \cap \widetilde{E}_j \quad \text{and} \quad y \in X \setminus \big(\bigcup_{i\in I} E_i\big) = X \setminus \big(\bigcup_{i\in I} \widehat{E}_i\big), \tag{107}$$

where the equality above is a consequence of condition *(d)* in the hypotheses. In particular $x \in \widetilde{E}_j$ and $y \in X \setminus \widehat{E}_j$ and, hence, based on *(a)* we have

$$\rho(x,y) \geq \mathrm{dist}_\rho(\widetilde{E}_j, X \setminus \widehat{E}_j) \geq c r_j \tag{108}$$

where, as before, $c > 0$ is the lower proportionality constant in (84). In this situation, using the definition of φ_j in (91), the first inequality in (90), (108), and the properties (51) and (56) of ω we may estimate

$$|\varphi_j(x) - \varphi_j(y)| = \varphi_j(x) = \frac{\psi_j(x)}{\Psi(x)} \leq \psi_j(x) \leq 1 \leq C\omega(r_j)^{-1}\omega(\rho(x,y)), \tag{109}$$

where $C > 0$ is a finite constant depending on β and the lower proportionality constant in (84). This proves the last case in the analysis of (97), finishing the proof of *(1)* in the conclusion of the theorem. The proof of Theorem 5.1 is now complete. □

There are several important instances when the hypotheses of Theorem 5.1 are satisfied. Yet, perhaps the most basic setting in which families of sets $\{E_j\}_{j\in I}$, $\{\widetilde{E}_j\}_{j\in I}$ and $\{\widehat{E}_j\}_{j\in I}$ satisfying the conditions hypothesized in Theorem 5.1 arise in a natural fashion is in relation to the Whitney decomposition of an open subset of a geometrically doubling quasi-metric space (for more details see the comment at the end of §6). In turn, this topic makes the object of the next section in the paper.

6. Whitney-type decompositions in geometrically doubling quasi-metric spaces.

A version of the classical Whitney decomposition theorem in the Euclidean setting (as presented in, e.g., [29, Theorem 1.1, p. 167]) has been worked out in [5, Theorem 3.1, p. 71] and [6, Theorem 3.2, p. 623] in the context of bounded open sets in spaces of homogeneous type. Recently, in [23], the scope of this work has been further refined as to apply to arbitrary open sets in a geometrically doubling quasi-metric space, equipped with a symmetric quasi-distance. Here we present a slight extension of this body of work by allowing quasi-distances which are not necessarily symmetric. Before formulating this result, in Theorem 6.2 below, we first define the class of geometrically doubling quasi-metric spaces.

Definition 6.1. A quasi-metric space (X, ρ) is called `geometrically doubling` if there exists a number $N \in \mathbb{N}$, called the `geometrically doubling constant of` (X, ρ), with the property that any ρ-ball of any given radius $r > 0$ in X may be covered by at most N ρ-balls in X of radii $r/2$.

Note that if (X, ρ) is a geometrically doubling quasi-metric space then

$$\begin{array}{c}\forall\theta \in (0,1)\exists N \in \mathbb{N} \text{ such that any } \rho\text{-ball of radius } r \text{ in } X \\ \text{may be covered by at most } N\ \rho\text{-balls in } X \text{ of radii } \theta r.\end{array} \tag{110}$$

In particular, this shows that if (X, ρ) is a geometrically doubling quasi-metric space and if $\rho' \approx \rho$ then (X, ρ') is also a geometrically doubling quasi-metric space.

The stage is set in order to discuss the main result in this section.

Theorem 6.2. *Let (X, ρ) be a geometrically doubling quasi-metric space. Then for each number $\lambda \in (1, +\infty)$ there exist constants $\Lambda \in (\lambda, +\infty)$ and $M \in \mathbb{N}$, both depending only on $C_\rho, \widetilde{C}_\rho$ as in (15)-(16), λ and the geometric doubling constant of (X, ρ), and which have the following significance.*

For each proper, nonempty, open subset $\mathcal{O}$ of the topological space (X, τ_ρ) there exist a sequence of points $\{x_j\}_{j\in\mathbb{N}}$ in $\mathcal{O}$ along with a family of real numbers $r_j > 0$, $j \in \mathbb{N}$, for which the following properties are valid:

(1) $\mathcal{O} = \bigcup_{j\in\mathbb{N}} B_\rho(x_j, r_j)$;

(2) $\sum_{j\in\mathbb{N}} \mathbf{1}_{B_\rho(x_j,\lambda r_j)} \leq M$ *on* $\mathcal{O}$. *In fact, there exists* $\varepsilon \in (0,1)$, *which depends only on* $C_\rho, \widetilde{C}_\rho, \lambda$ *and the geometrically doubling constant of* (X,ρ), *with the property that for any* $x_0 \in \mathcal{O}$

$$\#\Big\{j \in \mathbb{N} : B_\rho\big(x_0, \varepsilon \mathrm{dist}_\rho(x_0, X \setminus \mathcal{O})\big) \cap B_\rho(x_j, \lambda r_j) \neq \emptyset\Big\} \leq M. \tag{111}$$

(3) $B_\rho(x_j, \lambda r_j) \subseteq \mathcal{O}$ *and* $B_\rho(x_j, \Lambda r_j) \cap \big[X \setminus \mathcal{O}\big] \neq \emptyset$ *for every* $j \in \mathbb{N}$.

(4) $r_i \approx r_j$ *uniformly for* $i, j \in \mathbb{N}$ *such that* $B_\rho(x_i, \lambda r_i) \cap B_\rho(x_j, \lambda r_j) \neq \emptyset$.

Proof. In the case when the quasi-distance ρ is symmetric (i.e., when $\widetilde{C}_\rho = 1$), this result has been established in [23]. The present, slightly more general version considered here may be proved either by proceeding along similar lines, or by observing that the result in [23] self-improves to the current version as follows. Given ρ as in (13), consider its symmetrization ρ_{sym} defined in (25). Also, given $\lambda \in (1,\infty)$, consider a much larger dilation parameter $\lambda_{sym} > 1$. As noted earlier, (X, ρ_{sym}) is geometrically doubling and $\tau_\rho = \tau_{\rho_{sym}}$. Hence, by virtue of the result proved for symmetric quasi-distances from [23], we have the Whitney-like decomposition $\mathcal{O} = \bigcup_{j\in\mathbb{N}} B_{\rho_{sym}}(x_j, r_j)$ for some $\{x_j\}_{j\in\mathbb{N}} \subseteq \mathcal{O}$ and some real numbers $r_j > 0$, $j \in \mathbb{N}$. Then, since $\rho \leq \rho_{sym} \leq \widetilde{C}_\rho \rho$, it follows that for each $j \in \mathbb{N}$ we have $B_{\rho_{sym}}(x_j, r_j) \subseteq B_\rho(x_j, r_j)$ and $B_\rho(x_j, \lambda r_j) \subseteq B_{\rho_{sym}}(x_j, \lambda_{sym} r_j)$ if $\lambda_{sym} \geq \widetilde{C}_\rho \lambda$. It may then be verified without difficulty that $\mathcal{O} = \bigcup_{j\in\mathbb{N}} B_\rho(x_j, r_j)$ is a Whitney-like decomposition of $\mathcal{O}$ (in the sense described in the statement of the theorem). □

We conclude this section with a comment which sheds light on the connections between the theorem just presented and Theorem 5.1. Specifically, suppose $\mathcal{O}$ is a proper nonempty subset of a geometrically doubling quasi-metric space (X,ρ) and let $\lambda > 1$. Then Theorem 6.2 ensures the existence of a family $\{B_\rho(x_j, r_j)\}_{j\in\mathbb{N}}$ satisfying properties *(1)-(4)* above, for this choice of λ. If $\lambda' > 1$ is fixed with the property that $C_\rho < \lambda'$ and $\lambda' C_\rho < \lambda$ and we take the sets $E_j := B_\rho(x_j, r_j)$, $\widetilde{E}_j := B_\rho(x_j, \lambda' r_j)$ and $\widehat{E}_j := B_\rho(x_j, \lambda r_j)$, for each $j \in \mathbb{N}$, then conditions *(a)-(d)* in Theorem 5.1 are valid for the families $\{E_j\}_{j\in\mathbb{N}}$, $\{\widetilde{E}_j\}_{j\in\mathbb{N}}$, $\{\widehat{E}_j\}_{j\in\mathbb{N}}$ (with the radii r_j's playing the role of the parameters r_j's from the statement of Theorem 5.1).

7. **Linear extension operators preserving $\dot{\mathscr{C}}^\omega$ on geometrically doubling quasi-metric spaces.** In this section we formulate and prove the principal result of our paper. Concretely, under appropriate assumptions on ω, in Theorem 7.1 below we construct a linear, bounded extension operator mapping $\dot{\mathscr{C}}^\omega(E, \rho; \mathscr{V})$ into $\dot{\mathscr{C}}^\omega(X, \rho; \mathscr{V})$ where $\mathscr{V}$ is a quasi-normed vector space, and E is a subset of a geometrically doubling quasi-metric space (X,ρ). The latter condition (cf. Definition 6.1) is key in ensuring that the original design of such an extension operator, as envisioned by Whitney in [32] in the Euclidean setting, may be adapted to the present, considerably more general context, since it permits us to invoke our earlier results from §§5-6.

By way of comparison, there are several basic distinctions between the characters of Theorem 3.3 formulated in arbitrary quasi-metric spaces, on the one hand, and Theorem 7.1 formulated in geometrically doubling quasi-metric spaces, on the other hand. First, as already pointed out, the extension procedure in the latter theorem is linear, as opposed to the non-linear extension scheme used in the proof of the former theorem. However, another significant difference (which may be traced back to the conceptually different strategies employed in the proofs of the results in question) is the fact that Theorem 7.1 works for functions taking values in a possibly infinite dimensional vector space, whereas Theorem 3.3 can only be applied (componentwise) to functions taking values in a finite dimensional vector space.

Turning to specifics, we first make a couple of definitions. Call a nonempty subset $\mathscr{K}$ of a vector space $\mathscr{V}$ (over the reals) a `convex cone` provided

$$\lambda x \in \mathscr{K} \quad \forall x \in \mathscr{K} \quad \forall \lambda \in [0,+\infty), \quad \text{and} \quad x+y \in \mathscr{K} \quad \forall x,y \in \mathscr{K}. \tag{112}$$

Also, given an arbitrary nonempty subset $\mathscr{W}$ of a vector space $\mathscr{V}$, denote by $\mathrm{CC}(\mathscr{W})$ the smallest convex cone containing $\mathscr{W}$, i.e.,

$$\mathrm{CC}(\mathscr{W}) := \bigcap_{\substack{\mathscr{K}\, convex cone \\ \mathscr{W}\subseteq\mathscr{K}}} \mathscr{K}. \tag{113}$$

After this preamble, we are ready to state the principal result in this paper:

Theorem 7.1. *Let (X,ρ) be a geometrically doubling quasi-metric space and assume that E is a nonempty, closed subset of the topological space (X,τ_ρ). Fix a finite number $\beta \in (0,(\log_2 C_\rho)^{-1}]$, where C_ρ is as in (15), and consider a β-modulation ω which satisfies $\omega(0)=0$. Finally, assume that $(\mathscr{V},\|\cdot\|_{\mathscr{V}})$ is a quasi-normed vector space (over the reals).*

Then there exists an operator $\mathscr{E}$, mapping the vector space of $\mathscr{V}$-valued functions defined on E into the vector space of $\mathscr{V}$-valued functions defined on X, such that the following properties hold:

(1) $\mathscr{E}$ is linear and preserves constants (i.e., it maps constant functions on E into constant functions on X);

(2) $\mathscr{E}$ is an extension operator, in the sense that $(\mathscr{E}f)\big|_E = f$ for every function $f: E \to \mathscr{V}$;

(3) $\mathscr{E}$ is monotone, in the sense that $(\mathscr{E}f)(X) \subseteq \mathrm{CC}\big(f(E)\big)$ for every function $f: E \to \mathscr{V}$;

(4) $\mathscr{E}$ maps bounded functions on E into bounded functions on X, more precisely there exists a constant $c = c(\mathscr{V}) \in (0,+\infty)$ for which

$$\sup_{x\in X} \|(\mathscr{E}f)(x)\|_{\mathscr{V}} \le c \sup_{x\in E} \|f(x)\|_{\mathscr{V}}, \qquad \forall f: E \to \mathscr{V}; \tag{114}$$

(5) if $\#E \ge 2$, then $\mathscr{E}$ maps functions from $\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})$ into functions from $\dot{\mathscr{C}}^\omega(X,\rho;\mathscr{V})$, more precisely

$$\begin{gathered}\mathscr{E}: \dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V}) \longrightarrow \dot{\mathscr{C}}^\omega(X,\rho;\mathscr{V}) \text{ is a well-defined,} \\ \text{linear and bounded extension operator ;}\end{gathered} \tag{115}$$

(6) if ω is actually a β-modulus of continuity, then $\mathscr{E}$ maps continuous real-valued functions defined on E into continuous real-valued functions defined on X, i.e., $\mathscr{E}: \mathscr{C}^0(E;\mathscr{V}) \to \mathscr{C}^0(X;\mathscr{V})$ is well-defined and linear.

Finally, if the original hypotheses are strengthened to assuming that $(\mathscr{V},\|\cdot\|_{\mathscr{V}})$ is actually a quasi-Banach space and ω is actually a β-modulus of continuity, then *for any subset E of X (not necessarily closed) with $\#E \geq 2$ there exists a linear and bounded operator*

$$\begin{gathered}\mathscr{F} : \dot{\mathscr{C}}^{\omega}(E,\rho;\mathscr{V}) \longrightarrow \dot{\mathscr{C}}^{\omega}(X,\rho;\mathscr{V}) \\ \text{such that } (\mathscr{F}f)|_E = f, \quad \forall f \in \dot{\mathscr{C}}^{\omega}(E,\rho;\mathscr{V}).\end{gathered} \tag{116}$$

Before presenting the proof of Theorem 7.1 we find it useful to isolate a couple of auxiliary results. The first such auxiliary result is Aoki-Rolewicz's theorem; cf. [1], [16], [25], as well as the recent generalization in [23] which reads as follows:

Theorem 7.2. *Let $\mathscr{V}$ be a vector space equipped with a quasi-norm $\|\cdot\|_{\mathscr{V}}$. Then there exists a quasi-norm $\|\cdot\|_*$ on $\mathscr{V}$ which is equivalent to $\|\cdot\|_{\mathscr{V}}$ and which is a p-norm for some $p \in (0,1]$. More precisely, if $\kappa \in [2,+\infty)$ is as in (35) and if $p := (\log_2 \kappa)^{-1} \in (0,1]$, then a quasi-norm $\|\cdot\|_*$ may be constructed on $\mathscr{V}$ such that*

$$\begin{gathered}\kappa^{-2}\|x\|_{\mathscr{V}} \leq \|x\|_* \leq \|x\|_{\mathscr{V}} \quad \text{and} \\ \|x+y\|_*^p \leq \|x\|_*^p + \|y\|_*^p \text{ for all } x,y \in \mathscr{V}.\end{gathered} \tag{117}$$

The second auxiliary result needed in the proof of Theorem 7.1 is the extension procedure described in the lemma below.

Lemma 7.3. *Suppose that (X,ρ) is a quasi-metric space and recall the constants $C_\rho, \widetilde{C}_\rho \in [1,+\infty)$ from (15)-(16). Also, assume that $\beta \in (0, (\log_2 C_\rho)^{-1}]$ is a finite number and that ω is a β-modulus of continuity. Finally, fix a set $E \subseteq X$ of cardinality at least two, and suppose that the quasi-normed vector space $(\mathscr{V}, \|\cdot\|_{\mathscr{V}})$ is complete (i.e., is a quasi-Banach space). Then, with $\kappa \in [2,+\infty)$ as in (117),*

$$\begin{gathered}\text{for each } f \in \dot{\mathscr{C}}^{\omega}(E,\rho;\mathscr{V}) \text{there exists a unique function } \widetilde{f} \in \dot{\mathscr{C}}^{\omega}(\overline{E},\rho;\mathscr{V}) \\ \text{with the property that } \widetilde{f}|_E = f; \\ \text{in addition, } \widetilde{f} \text{ satisfies } \|\widetilde{f}\|_{\dot{\mathscr{C}}^{\omega}(\overline{E},\rho;\mathscr{V})} \leq \kappa^2 2^{\langle 2\beta \log_2 C_\rho\rangle + \langle \beta \log_2 \widetilde{C}_\rho\rangle} \|f\|_{\dot{\mathscr{C}}^{\omega}(E,\rho;\mathscr{V})},\end{gathered} \tag{118}$$

where $\overline{E}$ denotes the closure of E in the topology τ_ρ. Moreover,

$$\text{the mapping } \dot{\mathscr{C}}^{\omega}(E,\rho;\mathscr{V}) \ni f \mapsto \widetilde{f} \in \dot{\mathscr{C}}^{\omega}(\overline{E},\rho;\mathscr{V}) \text{ is a linear isomorphism.} \tag{119}$$

Proof. Let $\beta \in (0,+\infty)$ be as in the statement of the lemma and let the function ω be a β-modulation which vanishes continuously at the origin. In particular, by (iii) in Lemma 3.2, ω is continuous on $[0,+\infty)$. Finally, fix an arbitrary subset E of X with $\#E \geq 2$. Then, for each $f \in \dot{\mathscr{C}}^{\omega}(E,\rho;\mathscr{V})$, define

$$\begin{gathered}\widetilde{f} : \overline{E} \longrightarrow \mathscr{V}, \quad \widetilde{f}(x) := \lim_{j\to\infty} f(x_j) \text{ in } \mathscr{V}, \quad \text{if } x \in \overline{E}, \quad \text{and} \\ \{x_j\}_{j\in\mathbb{N}} \subseteq E \text{ is a sequence such that } \lim_{j\to\infty} \rho(x,x_j) = 0.\end{gathered} \tag{120}$$

We first make the claim that $\widetilde{f}$ is well-defined. That is, the limit in (120) exists, is finite, and does not depend on the choice of sequence. To see this pick an arbitrary $\varepsilon > 0$, fix $x \in \overline{E}$, and (given the nature of τ_ρ) select a sequence $\{x_j\}_{j\in\mathbb{N}} \subseteq E$ with the property $\lim_{j\to\infty} \rho(x,x_j) = 0$. Based on the fact that ω vanishes continuously at the origin, we may then find $\delta > 0$ such that $\omega(\delta) < \varepsilon$. For this δ, select $j_o \in \mathbb{N}$ large enough so that $\rho(x,x_j) < \delta$ whenever $j \in \mathbb{N}$ satisfies $j \geq j_o$. Granted this and the monotonicity of ω, we may write $\omega(\rho(x,x_j)) \leq \omega(\delta) < \varepsilon$ provided $j \in \mathbb{N}$ is such that $j \geq j_o$. In turn, if $x \in \overline{E}$ is as above, this allows us to estimate (by once again relying on the monotonicity and the slow-growth of ω),

$$\begin{aligned}\|f(x_j)-f(x_k)\|_{\mathscr{V}} &\leq \|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}\omega(\rho(x_j,x_k)) \\ &\leq \|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}\omega(C_\rho\max\{\rho(x_j,x),\rho(x,x_k)\}) \\ &\leq \|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}2^{\langle\beta\log_2 C_\rho\rangle}\omega(\max\{\rho(x_j,x),\rho(x,x_k)\}) \\ &= \|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}2^{\langle\beta\log_2 C_\rho\rangle}\max\Big\{\omega(\rho(x_j,x)),\omega(\rho(x,x_k))\Big\} \\ &\leq \|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}2^{\langle\beta\log_2 C_\rho\rangle}\max\Big\{\omega\big(\widetilde{C}_\rho\rho(x,x_j)\big),\omega(\rho(x,x_k))\Big\} \\ &< \varepsilon 2^{\langle\beta\log_2 C_\rho\rangle+\langle\beta\log_2\widetilde{C}_\rho\rangle}\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}, \end{aligned} \tag{121}$$

if $j,k\in\mathbb{N}$ are $\geq j_o$. This shows that $\{f(x_j)\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{V}$, hence convergent in $\mathscr{V}$, since this space is quasi-Banach. Having established this, the fact that $\widetilde{f}$ is unambiguously defined in (120) then follows by interlacing sequences. Going further, it is immediate that $\widetilde{f}$ is an extension of f since in the case when $x\in E$ we may take $\{x_j\}_{j\in\mathbb{N}}\subseteq E$ to be the constant sequence $x_j:=x$ for all $j\in\mathbb{N}$.

We now proceed to show that $\widetilde{f}\in\dot{\mathscr{C}}^\omega(\overline{E},\rho;\mathscr{V})$. To this end, fix $x,y\in\overline{E}$ and consider two sequences $\{x_j\}_{j\in\mathbb{N}},\{y_j\}_{j\in\mathbb{N}}$ of points in E with the property that $\lim\limits_{j\to\infty}\rho(x,x_j)=\lim\limits_{j\to\infty}\rho(y,y_j)=0$. Recall the quasi-norm $\|\cdot\|_*$ as in Theorem 7.2. Granted that this quasi-norm is continuous, (117), the definition of $\widetilde{f}$, Theorem 2.1, Lemma 3.2 and the fact that $f\in\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})$, we may estimate

$$\begin{aligned}\|\widetilde{f}(x)-\widetilde{f}(y)\|_{\mathscr{V}} &= \big\|\lim_{j\to\infty}f(x_j)-\lim_{j\to\infty}f(y_j)\big\|_{\mathscr{V}} \leq \kappa^2\big\|\lim_{j\to\infty}\big(f(x_j)-f(y_j)\big)\big\|_* \\ &\leq \kappa^2\limsup_{j\to\infty}\|f(x_j)-f(y_j)\|_* \leq \kappa^2\limsup_{j\to\infty}\|f(x_j)-f(y_j)\|_{\mathscr{V}} \\ &\leq \kappa^2\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}\limsup_{j\to\infty}\omega(\rho(x_j,y_j)) \\ &\leq \kappa^2\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}\limsup_{j\to\infty}\omega\big(C_\rho^2\rho_\#(x_j,y_j)\big) \\ &\leq \kappa^2 2^{\langle 2\beta\log_2 C_\rho\rangle}\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}\limsup_{j\to\infty}\omega(\rho_\#(x_j,y_j)) \\ &= \kappa^2 2^{\langle 2\beta\log_2 C_\rho\rangle}\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}\omega(\rho_\#(x,y)) \\ &\leq \kappa^2 2^{\langle 2\beta\log_2 C_\rho\rangle}\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}\omega\big(\widetilde{C}_\rho\rho(x,y)\big) \\ &\leq \kappa^2 2^{\langle 2\beta\log_2 C_\rho\rangle+\langle\beta\log_2\widetilde{C}_\rho\rangle}\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}\omega(\rho(x,y)), \end{aligned} \tag{122}$$

where the second equality is a result of the continuity of both ω (as pointed out earlier) and $\rho_\#$ (cf. (28)), and where the monotonicity and slow-growth property of ω have been used repeatedly. The above argument shows that $\widetilde{f}\in\dot{\mathscr{C}}^\omega(\overline{E},\rho;\mathscr{V})$ with $\|\widetilde{f}\|_{\dot{\mathscr{C}}^\omega(\overline{E},\rho;\mathscr{V})}\leq\kappa^2 2^{\langle 2\beta\log_2 C_\rho\rangle+\langle\beta\log_2\widetilde{C}_\rho\rangle}\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}$, as desired.

In order to establish the uniqueness aspect of the claim in (118) assume in addition to $\widetilde{f}$, there exists another function $g\in\dot{\mathscr{C}}^\omega(\overline{E},\rho;\mathscr{V})$ such that $g|_E=f$. Then for any point $x\in\overline{E}$ and any sequence $\{x_j\}_{j\in\mathbb{N}}\subseteq E$ with the property $\lim\limits_{j\to\infty}\rho(x,x_j)=0$ we have $\widetilde{f}(x)=\lim\limits_{j\to\infty}f(x_j)=\lim\limits_{j\to\infty}g(x_j)=g(x)$. Hence, $\widetilde{f}=g$ on $\overline{E}$, as wanted.

Let us now prove that the map $\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})\ni f\mapsto\widetilde{f}\in\dot{\mathscr{C}}^\omega(\overline{E},\rho;\mathscr{V})$ is linear. As a consequence of the above analysis, if the functions $f,g\in\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})$ then

$\widetilde{f+g}, \widetilde{f}+\widetilde{g} \in \dot{\mathscr{C}}^{\omega}(\overline{E},\rho;\mathscr{V})$ satisfy $\widetilde{f+g}|_E = f+g = (\widetilde{f}+\widetilde{g})|_E$. Based on this and the uniqueness property just established, we may then conclude that $\widetilde{f+g} = \widetilde{f}+\widetilde{g}$. Hence, the map in question is additive. Its homogeneity may also be established using the same pattern of reasoning. From this, the linearity of the mapping in question follows. Obviously, this mapping is one-to-one. Since for every function $g \in \dot{\mathscr{C}}^{\omega}(\overline{E},\rho;\mathscr{V})$ we have $\widetilde{g|_E} = g$, we see that this mapping is also onto hence, ultimately, a linear isomorphism. This completes the proof of Lemma 7.3. □

We are now ready to present the proof of Theorem 7.1. Compared with the proof of Theorem 3 on pp. 174-175 in [29], corresponding to the version of our result formulated for real-valued functions defined on subsets of the Euclidean space, our argument relies on appropriate substitutes for the tools listed in [i]-[iii] from §1, in more general, non-Euclidean settings.

Proof of Theorem 7.1. Assume that (X,ρ) is a geometrically doubling quasi-metric space and fix an arbitrary, nonempty, closed subset E of (X,τ_ρ). Also, suppose that $\beta \in (0,+\infty)$ is as in the statement of the theorem and that ω is a β-modulation satisfying $\omega(0)=0$. If $E=X$, we simply take $\mathscr{E}$ to be the identity operator, so we assume in what follows that $E \neq X$. In this case, $X\setminus E$ is a proper open subset of (X,τ_ρ).

Going further, pick a constant $\lambda > C_\rho^2$ and consider the Whitney decomposition $X\setminus E = \bigcup_{j\in\mathbb{N}} B_\rho(x_j,r_j)$ as given by Theorem 6.2. Let $\Lambda \in (\lambda,+\infty)$ and $M\in\mathbb{N}$ be as in the conclusion of Theorem 6.2. Next, select a constant $\lambda' \in \big(C_\rho, \lambda/C_\rho\big)$ and define the families $\{E_j\}_{j\in\mathbb{N}}$, $\{\widetilde{E}_j\}_{j\in\mathbb{N}}$, $\{\widehat{E}_j\}_{j\in\mathbb{N}}$ by setting for each $j\in\mathbb{N}$

$$E_j := B_\rho(x_j,r_j), \quad \widetilde{E}_j := B_\rho(x_j,\lambda' r_j) \text{ and } \widehat{E}_j := B_\rho(x_j,\lambda r_j). \tag{123}$$

Then, as indicated in the discussion at the end of §5, the hypotheses of Theorem 5.1 are satisfied for this choice of families. Next, Theorem 5.1 (employed for the given β-modulation ω) yields a partition of unity $\{\varphi_j\}_{j\in\mathbb{N}}$ satisfying properties *(1)-(3)* listed in the conclusion of this result. Note that, for each $j\in\mathbb{N}$ it is possible to choose a point $p_j\in E$ with the property that

$$\tfrac{1}{2}\mathrm{dist}_\rho\big(p_j, B_\rho(x_j,\lambda' r_j)\big) \le \mathrm{dist}_\rho\big(E, B_\rho(x_j,\lambda' r_j)\big) \le \mathrm{dist}_\rho\big(p_j, B_\rho(x_j,\lambda' r_j)\big). \tag{124}$$

Hence, since as a consequence of $\{B_\rho(x_j,r_j)\}_{j\in\mathbb{N}}$ being the balls in the Whitney decomposition of $X\setminus E$,

$$\mathrm{dist}_\rho\big(E, B_\rho(x_j,\lambda' r_j)\big) \approx r_j, \qquad \text{uniformly in } j\in\mathbb{N}, \tag{125}$$

it follows from this and (124) that

$$\mathrm{dist}_\rho\big(p_j, B_\rho(x_j,\lambda' r_j)\big) \approx r_j, \qquad \text{uniformly in } j\in\mathbb{N}. \tag{126}$$

Given an arbitrary function $f: E\to\mathscr{V}$, we then proceed to define

$$(\mathscr{E}f)(x) := \begin{cases} f(x) & \text{if } x\in E, \\ \sum\limits_{j\in\mathbb{N}} f(p_j)\varphi_j(x) & \text{if } x\in X\setminus E, \end{cases} \tag{127}$$

and note that, in light of (86) and *(2)* in the conclusion of Theorem 6.2, we have that $\mathscr{E}f : X\to\mathscr{V}$ is a well-defined function. Then properties *(1)-(3)* in the statement of the theorem are direct consequences of (127). As far as *(4)* is concerned, we start by recalling the constants $\kappa\in[2,+\infty)$, $p=(\log_2\kappa)^{-1}\in(0,1]$ as well as the quasi-norm $\|\cdot\|_*$ from Theorem 7.2. Then, from (127) and (117) we see that for every $x \in X \setminus E$

$$\begin{aligned} \|(\mathscr{E}f)(x)\|_{\mathscr{V}}^p &\leq \kappa^{2p}\|(\mathscr{E}f)(x)\|_*^p \leq \kappa^{2p}\sum_{j\in\mathbb{N}}\|f(p_j)\|_*^p\varphi_j(x) \\ &\leq \kappa^{2p}\sum_{j\in\mathbb{N}}\|f(p_j)\|_{\mathscr{V}}^p\varphi_j(x) \\ &\leq \kappa^{2p}\sup_{y\in E}\|f(y)\|_{\mathscr{V}}^p\sum_{j\in\mathbb{N}}\varphi_j(x) = \kappa^{2p}\Big(\sup_{y\in E}\|f(y)\|_{\mathscr{V}}\Big)^p, \end{aligned} \tag{128}$$

from which (114) readily follows (with $c := \kappa^2$).

Moving on, assume that $\#E \geq 2$, with the goal of proving that the operator

$$\mathscr{E} : \dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V}) \longrightarrow \dot{\mathscr{C}}^\omega(X,\rho;\mathscr{V}) \quad \text{is well-defined and bounded.} \tag{129}$$

Hence, we seek to show that there exists a finite constant $C \geq 0$ with the property that for any $f \in \dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})$ there holds

$$\|(\mathscr{E}f)(x) - (\mathscr{E}f)(y)\|_{\mathscr{V}} \leq C\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}\omega(\rho(x,y)), \quad \forall x,y \in X. \tag{130}$$

Obviously, the estimate in (130) holds if $C \geq 1$ whenever $x, y \in E$. Consider next the case when $x \in X \setminus E$ and $y \in E$. As a preliminary matter, we claim that

$$j \in \mathbb{N} \text{ and } x \in B_\rho(x_j, \lambda' r_j) \Longrightarrow \rho(x,p_j) \approx r_j, \tag{131}$$

with proportionality constant depending only on ρ. To justify the claim in (131), note that if $x \in B_\rho(x_j, \lambda r_j)$ for some $j \in \mathbb{N}$ then

$$\rho(x,z) \leq C_\rho \max\{\rho(x,x_j),\rho(x_j,z)\} < \lambda C_\rho r_j, \qquad \forall z \in B_\rho(x_j,\lambda r_j), \tag{132}$$

hence, further, for every $z \in B_\rho(x_j, \lambda r_j)$,

$$\rho(x,p_j) \leq C_\rho \max\{\rho(x,z),\rho(z,p_j)\} < C_\rho \max\{\lambda C_\rho r_j, \widetilde{C}_\rho \rho(p_j,z)\}. \tag{133}$$

Taking the infimum over all $z \in B_\rho(x_j, \lambda r_j)$ and keeping in mind (126) we therefore arrive at the conclusion that

$$\begin{aligned} \rho(x,p_j) &\leq C_\rho \max\big\{\lambda C_\rho r_j, \widetilde{C}_\rho \mathrm{dist}_\rho\big(p_j, B_\rho(x_j,\lambda r_j)\big)\big\} \\ &\leq C_\rho \max\big\{\lambda C_\rho r_j, \widetilde{C}_\rho \mathrm{dist}_\rho\big(p_j, B_\rho(x_j,\lambda' r_j)\big)\big\} \\ &\leq C r_j, \end{aligned} \tag{134}$$

for some $C = C(\rho) \in (0,+\infty)$. In summary, this analysis shows that there exists $C = C(\rho) \in (0,+\infty)$ for which

$$j \in \mathbb{N} \text{ and } x \in B_\rho(x_j, \lambda r_j) \Longrightarrow \rho(x,p_j) \leq C r_j, \tag{135}$$

which is a slightly stronger version than what is really needed in (131) (however, this will be useful later on). In the opposite direction, if $x \in B_\rho(x_j, \lambda' r_j)$ for some $j \in \mathbb{N}$ then by appealing once more to (126) we may write

$$\rho(x,p_j) \geq (\widetilde{C}_\rho)^{-1}\rho(p_j,x) \geq (\widetilde{C}_\rho)^{-1}\mathrm{dist}_\rho\big(p_j, B_\rho(x_j,\lambda' r_j)\big) \geq c r_j, \tag{136}$$

for some $c = c(\rho) \in (0,+\infty)$. This concludes the proof of (131). As a consequence of (131) and (125) we then obtain

$$\rho(x,p_j) \approx \mathrm{dist}_\rho\big(E, B_\rho(x_j,\lambda' r_j)\big), \text{ uniformly in } j \in \mathbb{N} \text{ and } x \in B_\rho(x_j,\lambda' r_j). \tag{137}$$

Going further, whenever $y \in E$ and $x \in B_\rho(x_j, \lambda' r_j)$ for some $j \in \mathbb{N}$, (137) allows us to estimate

$$\rho(y, p_j) \le C_\rho \max\{\rho(y,x), \rho(x,p_j)\} \le C \max\big\{\rho(y,x), \mathrm{dist}_\rho\big(E, B_\rho(x_j, \lambda' r_j)\big)\big\} \le C\rho(y,x). \tag{138}$$

Hence, for some finite $C = C(\rho) > 0$, independent of x, y, j, we have

$$y \in E, j \in \mathbb{N} \text{ and } x \in B_\rho(x_j, \lambda' r_j) \Longrightarrow \rho(y, p_j) \le C\rho(x,y). \tag{139}$$

Based on (117), (139), (56), the fact that $f \in \dot{\mathscr{C}}^\omega(E, \rho; \mathscr{V})$ and the properties of the functions $\{\varphi_j\}_{j\in\mathbb{N}}$, whenever $x \in X \setminus E$ and $y \in E$ we may therefore estimate (using the monotonicity and the slow-growth property (56) of the function ω),

$$\begin{aligned}
\|(\mathscr{E}f)(y) - (\mathscr{E}f)(x)\|_{\mathscr{V}}^p &= \Big\| f(y) - \sum_{j\in\mathbb{N}} f(p_j)\varphi_j(x)\Big\|_{\mathscr{V}}^p = \Big\|\sum_{j\in\mathbb{N}} (f(y) - f(p_j))\varphi_j(x)\Big\|_{\mathscr{V}}^p \\
&\le \kappa^{2p} \Big\|\sum_{j\in\mathbb{N}} (f(y) - f(p_j))\varphi_j(x)\Big\|_*^p \\
&\le \kappa^{2p} \sum_{j\in\mathbb{N}} \|f(y) - f(p_j)\|_*^p \varphi_j(x)^p \\
&\le \kappa^{2p} \sum_{j\in\mathbb{N}} \|f(y) - f(p_j)\|_{\mathscr{V}}^p \varphi_j(x)^p \\
&\le \kappa^{2p} \sum_{\substack{j\in\mathbb{N} \text{ such that} \\ x \in B_\rho(x_j, \lambda' r_j)}} \|f(y) - f(p_j)\|_{\mathscr{V}}^p \varphi_j(x)^p \\
&\le C\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}^p \sum_{\substack{j\in\mathbb{N} \text{ such that} \\ x \in B_\rho(x_j, \lambda' r_j)}} \omega(\rho(y, p_j))^p \varphi_j(x)^p \\
&\le C\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}^p \omega(\rho(x,y))^p,
\end{aligned} \tag{140}$$

since $0 \le \varphi_j \le 1$ for every $j \in \mathbb{N}$, and since by *(2)* in Theorem 6.2

$$\text{the cardinality of } \{j \in \mathbb{N} : x \in B_\rho(x_j, \lambda' r_j)\} \text{ is } \le M. \tag{141}$$

Of course, estimate (140) suits our purposes. The situation when $y \in X \setminus E$ and $x \in E$ is handled similarly, so there remains to treat the case when $x, y \in X \setminus E$, which we now consider. We shall investigate two separate subcases, starting with:

`Subcase I:` *Assume that the points $x, y \in X \setminus E$ are such that*

$$\rho(x,y) < \varepsilon \mathrm{dist}_\rho(x, E) \quad \textit{where} \quad 0 < \varepsilon < \frac{\lambda}{C_\rho(\Lambda \widetilde{C}_\rho C_\rho^2 + \lambda)}. \tag{142}$$

The relevance of the choice made for ε will become more apparent later. For now, we wish to mention that such a choice forces $\varepsilon \in (0, 1/C_\rho)$. To get started in earnest, we make the claim that in the above scenario, we have

$$\mathrm{dist}_\rho(x, E) \le \Big(\frac{C_\rho}{1 - \varepsilon C_\rho}\Big) \mathrm{dist}_\rho(y, E). \tag{143}$$

Indeed, for every $z \in E$ we may write

$$\mathrm{dist}_\rho(x,E) \le \rho(x,z) \le C_\rho\big(\rho(x,y) + \rho(y,z)\big) \le C_\rho\big(\varepsilon \mathrm{dist}_\rho(x,E) + \rho(y,z)\big), \tag{144}$$

hence $(1-\varepsilon C_\rho)\mathrm{dist}_\rho(x,E) \le C_\rho \rho(y,z)$. Taking the infimum over all $z \in E$, (143) follows. Moving on, observe that

$$(\mathscr{E}f)(x) - (\mathscr{E}f)(y) = \sum_{j\in\mathbb{N}} (f(p_j) - f(z))(\varphi_j(x) - \varphi_j(y)), \qquad \forall z \in E. \tag{145}$$

Choose now $z \in E$ such that

$$\frac{1}{2}\rho(x,z) \le \mathrm{dist}_\rho(x,E) \le \rho(x,z) \tag{146}$$

and note that this forces $\rho(x,z) \approx \mathrm{dist}_\rho(x,E) \le \rho(x,p_j)$. In concert with (135), this implies

$$j \in \mathbb{N} \text{ and } x \in B_\rho(x_j, \lambda r_j) \Longrightarrow \rho(p_j, z) \le C_\rho \max\{\rho(p_j,x), \rho(x,z)\} \le C r_j, \tag{147}$$

for some $C = C(\rho) \in (0,+\infty)$. Having established (147), we next write formula (145) for $z \in E$ as in (146) and make use of (117), the fact that $f \in \dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})$, along with the properties of $\{\varphi_j\}_{j\in\mathbb{N}}$, in order to estimate

$$\begin{aligned}
\|(\mathscr{E}f)(x) - (\mathscr{E}f)(y)\|_{\mathscr{V}}^p &\le \kappa^{2p} \|(\mathscr{E}f)(x) - (\mathscr{E}f)(y)\|_*^p \\
&\le \kappa^{2p} \sum_{j\in\mathbb{N}} \|f(p_j) - f(z)\|_*^p |\varphi_j(x) - \varphi_j(y)|^p \\
&\le \kappa^{2p} \sum_{j\in\mathbb{N}} \|f(p_j) - f(z)\|_{\mathscr{V}}^p |\varphi_j(x) - \varphi_j(y)|^p \\
&\le C \|f\|^p_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})} \omega(\rho(x,y))^p \Big\{ \sum_{j\in\mathbb{N}} \omega(\rho(p_j,z))^p \\
&\qquad \times \|\varphi_j\|^p_{\dot{\mathscr{C}}^\omega(X,\rho)} \big[\mathbf{1}_{B_\rho(x_j,\lambda' r_j)}(x) + \mathbf{1}_{B_\rho(x_j,\lambda' r_j)}(y)\big] \Big\} \\
&\le C \|f\|^p_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})} \omega(\rho(x,y))^p \big(A_x + A_y\big),
\end{aligned} \tag{148}$$

where we have set

$$\begin{aligned}
A_x &:= \sum_{\substack{j\in\mathbb{N} \text{ such that} \\ x\in B_\rho(x_j,\lambda' r_j)}} \Big[\omega(\rho(p_j,z))\omega(r_j)^{-1}\Big]^p, \\
A_y &:= \sum_{\substack{j\in\mathbb{N} \text{ such that} \\ y\in B_\rho(x_j,\lambda' r_j)}} \Big[\omega(\rho(p_j,z))\omega(r_j)^{-1}\Big]^p.
\end{aligned} \tag{149}$$

Now, (141), (147), the monotonicity of ω and (56) give that $A_x \le C$, for some finite constant $C = C(\rho,\beta) \ge 0$. In order to derive a similar estimate for A_y, assume that

$$j \in \mathbb{N} \text{ is such that } y \in B_\rho(x_j, \lambda' r_j). \tag{150}$$

Then by (143), (150), and the fact that $B_\rho(x_j, \Lambda r_j) \cap E \ne \emptyset$ we have

$$\mathrm{dist}_\rho(x,E) \le \Big(\frac{C_\rho}{1-\varepsilon C_\rho}\Big)\mathrm{dist}_\rho(y,E) \le \Lambda\Big(\frac{C_\rho^2}{1-\varepsilon C_\rho}\Big) r_j. \tag{151}$$

In turn, (151) permits us to deduce that

$$\begin{aligned}
\rho(x_j,x) &\le C_\rho \max\{\rho(x_j,y), \rho(y,x)\} \le C_\rho \max\Big\{\lambda' r_j, \varepsilon \widetilde{C}_\rho \mathrm{dist}_\rho(x,E)\Big\} \\
&\le C_\rho r_j \max\Big\{\lambda', \varepsilon \widetilde{C}_\rho \Lambda\Big(\tfrac{C_\rho^2}{1-\varepsilon C_\rho}\Big)\Big\} \\
&< \lambda r_j,
\end{aligned} \tag{152}$$

where the last inequality is a consequence of the fact that $\lambda' C_\rho < \lambda$ and the way ε has been chosen in (142). Estimate (152) shows that

$$\text{if } j \text{ is as in (150) then } x \in B_\rho(x_j, \lambda r_j). \tag{153}$$

With (153) in hand, a reference to (147) then gives

$$\text{if } j \text{ is as in (150) then } \rho(p_j, z) \leq C r_j \text{ whenever } z \text{ is as in (146)}, \tag{154}$$

for some finite $C = C(\rho) > 0$. Having proved (154) then the estimate $A_y \leq C$ for some $C = C(\rho, \beta) < +\infty$ follows as in the case of A_x, already treated. Altogether, this proves that $A_x + A_y \leq C = C(\rho, \beta) < +\infty$ which, in combination with (148), shows that $\|(\mathscr{E}f)(x) - (\mathscr{E}f)(y)\|_{\mathscr{V}} \leq C\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}\omega(\rho(x,y))$, under the hypotheses specified in Subcase I. This bound is of the right order, and this completes the treatment of Subcase I.

`Subcase II:` *With the parameter $\varepsilon > 0$ as in Subcase I, assume that $x, y \in X \setminus E$ are such that*

$$\rho(x, y) \geq \varepsilon \mathrm{dist}_\rho(x, E). \tag{155}$$

Consider a point $z \in E$ as in (146) and note that (155) forces

$$\rho(x, z) \leq 2\mathrm{dist}_\rho(x, E) \leq 2\varepsilon^{-1}\rho(x, y). \tag{156}$$

Hence, we also have $\rho(z, y) \leq C_\rho \max\{\rho(z, x), \rho(x, y)\} \leq C\rho(x, y)$ for some constant $C = C(\rho, \varepsilon) \in (0, +\infty)$. Consequently, based on this estimate, (35), the monotonicity and slow-growth property of ω, we deduce that

$$\begin{aligned}\|(\mathscr{E}f)(x) - (\mathscr{E}f)(y)\|_{\mathscr{V}} &\leq C\|(\mathscr{E}f)(x) - (\mathscr{E}f)(z)\|_{\mathscr{V}} + C\|(\mathscr{E}f)(z) - (\mathscr{E}f)(y)\|_{\mathscr{V}} \\ &\leq C\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}\omega(\rho(x,z)) + C\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}\omega(\rho(z,y)) \\ &\leq C\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}\omega(\rho(x,y))\end{aligned} \tag{157}$$

for some constant $C = C(\rho, \varepsilon, \beta, \kappa) \in (0, +\infty)$, by what we have established in the first part of the proof (i.e., using (140) twice, once for $x \in X \setminus E$ and $z \in E$ and, a second time, for $y \in X \setminus E$ and $z \in E$). This completes the treatment of the situation considered in Subcase II.

In summary, the above analysis proves that there exists $C = C(\rho, \beta, \varepsilon, \kappa) > 0$, a finite constant, with the property that for every $f \in \dot{\mathscr{C}}^\omega(E, \rho)$ we have

$$\|(\mathscr{E}f)(x) - (\mathscr{E}f)(y)\|_{\mathscr{V}} \leq C\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho;\mathscr{V})}\omega(\rho(x,y)), \qquad \forall x, y \in X. \tag{158}$$

This shows that the operator (115) is well-defined and bounded (recall that its linearity has already been noted), completing the proof of item *(5)* in the statement of the theorem.

Moving on to item *(6)*, the goal becomes showing that the operator $\mathscr{E}$ defined in (127) has the property that

$$\mathscr{E}f : X \to \mathscr{V} \text{ is continuous whenever } f : E \to \mathscr{V} \text{ is continuous}, \tag{159}$$

provided ω is actually a β-modulus of continuity. To this end, suppose ω is a β-modulation which vanishes continuously at the origin (recall that this forces ω to be continuous on $[0, +\infty)$; cf. *(iii)* in Lemma 3.2). Also, fix an arbitrary continuous function $f : E \to \mathscr{V}$. Note that, by design, $\mathscr{E}f$ is continuous on the open set $X \setminus E$ (since the sum in (127) is locally finite and the φ_j's are continuous as a result of (43)). There remains to show that $\mathscr{E}f$ is continuous at any point in E. Furthermore, since (as seen from Theorem 2.1) the topology τ_ρ is metrizable, it suffices to use the sequential characterization of continuity. Fix $z \in E$ and assume that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of points in X which converges to z in the topology τ_ρ. Introduce $N_0 := \{n \in \mathbb{N} : x_n \in E\}$ and $N_1 := \{n \in \mathbb{N} : x_n \in X \setminus E\}$. Then, on the one hand,

$$\lim_{N_0 \ni n\to\infty} (\mathscr{E} f)(x_n) = \lim_{N_0 \ni n\to\infty} f(x_n) = f(z) \quad \text{in } \mathscr{V}, \tag{160}$$

since f is continuous on E. On the other hand, for each $n \in N_1$, much as in (140) we may estimate

$$\|(\mathscr{E} f)(x_n) - (\mathscr{E} f)(z)\|^p_{\mathscr{V}} \leq C \sum_{\substack{j\in\mathbb{N} \text{ such that} \\ x_n\in B_\rho(x_j,\lambda' r_j)}} \|f(z) - f(p_j)\|^p_{\mathscr{V}}. \tag{161}$$

Let us also note that the version of (139) in the notation currently employed reads

$$j \in \mathbb{N} \text{ and } x_n \in B_\rho(x_j, \lambda' r_j) \Longrightarrow \rho(z, p_j) \leq C\rho(x_n, z), \tag{162}$$

for some finite $C = C(\rho) > 0$, independent of $n \in N_1$. Fix an arbitrary $\varepsilon > 0$ and, based on the continuity of f at z, pick $\delta > 0$ with the property that

$$\|f(z) - f(w)\|_{\mathscr{V}} < \varepsilon \text{ whenever } w \in E \text{ is such that } \rho(z, w) < \delta. \tag{163}$$

Since $\lim_{N_1\ni n\to\infty} x_n = z$, it follows that there exists $m \in \mathbb{N}$ such that

$$\rho(x_n, z) < \delta/C \text{ for each } n \in N_1 \text{ with the property that } n \geq m, \tag{164}$$

where the constant C is as in (162). Thus,

$$\|(\mathscr{E} f)(x_n) - (\mathscr{E} f)(z)\|_{\mathscr{V}} \leq (CM)^{1/p}\varepsilon, \quad \text{for every } n \in N_1 \text{ with } n \geq m, \tag{165}$$

by (161), (162), (163), and (141). Since $\varepsilon > 0$ was arbitrary, it follows from (160) and (165) that $\mathscr{E} f$ is continuous at z. This completes the justification of (159), and finishes the proof of item *(6)*.

At this stage, we are left with dealing with the last claim in the statement of the theorem, pertaining to the existence of an extension operator as in (116) when E is an arbitrary subset of X with $\#E \geq 2$, in the case when ω is a β-modulus of continuity and the space $(\mathscr{V}, \|\cdot\|_{\mathscr{V}})$ is quasi-Banach. In such a scenario, we invoke Theorem 7.1 for the set closed set $\overline{E}$ in order to obtain a bounded linear extension operator $\mathscr{E} : \dot{\mathscr{C}}^\omega(\overline{E}, \rho; \mathscr{V}) \to \dot{\mathscr{C}}^\omega(X, \rho; \mathscr{V})$, then for each $f \in \dot{\mathscr{C}}^\omega(E, \rho; \mathscr{V})$ define $\mathscr{F} f := \mathscr{E}(\widetilde{f})$, where $\widetilde{f}$ is as in (118). Granted (119) in Lemma 7.3, the desired conclusion follows. This finishes the proof of Theorem 7.1. □

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