TOPICS IN HARMONIC ANALYSIS AND PARTIAL DIFFERENTIAL EQUATIONS: EXTENSION THEOREMS AND GEOMETRIC MAXIMUM PRINCIPLES

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The undersigned, appointed by the Dean of the Graduate School, have examined the thesis entitled

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a candidate for the degree of Master of Arts and hereby certify that in their opinion it is worthy of acceptance.

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ABSTRACT

The present thesis consists of two main parts. In the first part, we prove that a function defined on a closed subset of a geometrically doubling quasi-metric space which satisfies a Hölder-type condition may be extended to the entire space with preservation of regularity. The proof proceeds along the lines of the original work of Whitney in 1934 and yields a linear extension operator. A similar extension result is also proved in the absence of the geometrically doubling hypothesis, albeit the resulting extension procedure is nonlinear in this case. The results presented in this part are based upon work done in [8] in collaboration M. Mitrea.

In the second part of the thesis we prove that an open, proper, nonempty subset of $\mathbb{R}^n$ is a locally Lyapunov domain if and only if it satisfies a uniform hour-glass condition. The latter is a property of a purely geometrical nature, which amounts to the ability of threading the boundary, at any location, in between the two rounded components (referred to as pseudo-balls) of a certain fixed region, whose shape resembles that of an ordinary hour-glass, suitably re-positioned. The limiting cases of the result (pertaining to the curvature of the hour-glass at the contact point) are as follows: Lipschitz domains may be characterized by a uniform double cone condition, whereas domains of class $\mathscr{C}^{1,1}$ may be characterized by a uniform two-sided ball condition. Additionally, we discuss a sharp generalization of the Hopf-Oleinik Boundary Point Principle for domains satisfying a one-sided, interior pseudo-ball condition, for semi-elliptic operators with singular drift. This, in turn, is used to obtain a sharp version of Hopf's Strong Maximum Principle for second-order, non-divergence

form differential operators with singular drift. This part of my thesis originates from a recent paper in collaboration with D. Brigham, V. Maz'ya, M. Mitrea, and E. Ziadé [9].

Chapter 1

Introduction

This thesis consists of two distinct yet related parts. A common feature that both parts share is the emphasis on the geometrical aspects of the problem. In the first part, we address under what minimal geometrical conditions are we gauranteed an operator which extends functions with preservation of smoothness. In the second part of this thesis, we characterize various classes of domains in terms which are purely geometric. In turn, this allows us to investigate how the geometry of a given domain affects the nature of solutions to various partial differential equations. The work in Part I of my thesis is based on an article in partnership with M. Mitrea [8]. While the results in Part I are based on joint work with Mitrea, the second part originates from a recent paper in collaboration with D. Brigham, V. Maz'ya, M. Mitrea, and E. Ziadé [9].

Over the years, results pertaining to extending classes of functions satisfying certain regularity properties (e.g. continuity, Lipschitzianty) from a subset of a space to the entire space while retaining the regularity properties have played a basic role in analysis. As is well known, a continuous function defined on a closed subset of a metric space may be extended with preservation of continuity to the entire space (see, e.g., [19, Exercise 4.1.F]). Indeed if E is a closed subset of a metric space (X, d) and $f : E \to \mathbb{R}$ is a continuous function then a concrete formula for a continuous extension of f to

X is given by Hausdorff's formula

$$F(x) := \begin{cases} \inf\left\{f(y) + \frac{d(x,y)}{\mathrm{dist}_d(x,E)} - 1 : \, y \in E\right\} & \text{if } \, x \in X \setminus E, \\ f(x) & \text{if } \, x \in E \end{cases} \tag{1.1}$$

where for $x \in X$ we define $\mathrm{dist}_d(x,E) := \inf\{d(x,y) : \, y \in E\}$. Although for each fixed set E, the recipe presented in (1.1) yields a concrete formula for the extension function, the operator $f \mapsto F$ is nonlinear. However this is an issue that was resolved , under suitable background assumptions on the ambient, in the proceeding years (cf. [91], on which we shall comment more on later).

Regarding the extension of functions with higher regularity properties we wish to mention the pioneering work of E.J. McShane [68], H. Whitney [91] and M.D. Kirszbraun [58]. The main result in [68] is to the effect that if (X,d) is a metric space and if $E \subseteq X$ is an arbitrary nonempty set then for any Lipschitz function $g \in \mathrm{Lip}\,(E,d)$ there exists $f \in \mathrm{Lip}\,(X,d)$ such that $f|_E = g$ and $\|g\|_{\mathrm{Lip}\,(E,d)} = \|f\|_{\mathrm{Lip}\,(X,d)}$. Indeed, in [68], E.J. McShane took f to be either

$$f^*(x) := \sup\left\{g(y) - \|g\|_{\mathrm{Lip}\,(E,d)}\, d(x,y) : \, y \in E\right\}, \qquad \forall\, x \in X \tag{1.2}$$

or similarly,

$$f_*(x) := \inf\left\{g(y) + \|g\|_{\mathrm{Lip}\,(E,d)}\, d(x,y) : \, y \in E\right\}, \qquad \forall\, x \in X. \tag{1.3}$$

In fact, the extension functions constructed in (1.2) and (1.3) are maximal and minimal (respectively) in the following sense. If $f \in \mathrm{Lip}\,(X,d)$ such that $f|_E = g$ and $\|g\|_{\mathrm{Lip}\,(E,d)} = \|f\|_{\mathrm{Lip}\,(X,d)}$ then $f^* \leq f \leq f_*$ on X. However as before with (1.1), the extension operators $g \mapsto f^*$and $g \mapsto f_*$ are nonlinear. One of the important aspects of this result is that it is applicable to any nonempty subset of a general metrizable space.

Concerning the non-linearity character of McShane's extension , in 1934, H. Whitney succeeded in constructing a linear extension operator in the Euclidean setting, which also preserves higher degrees of smoothness. Somewhat more specifically, in [91], Whitney gave necessary and sufficient conditions on an array of functions $\{f^\alpha\}_{|\alpha|\leq m}$ defined on a closed subset E of $\mathbb{R}^n$ ensuring the existence of a functions $F \in C^m(\mathbb{R}^m)$ with the property that $(\partial^\alpha F)|_E = f^\alpha$ whenever $|\alpha| \leq m$. In addition, Whitney's extension operator $\{f^\alpha\}_{|\alpha|\leq m} \mapsto F$ is universal (in the sense that it simultaneously preserves all orders of smoothness), as well as linear. A timely exposition of this result may be found in [87]. The proof presented there makes use of three basic ingredients, namely:

1. the existence of a Whitney decomposition of an open set (into Whitney balls of bounded overlap),

2. the existence of a smooth partition of unity subordinate (in an appropriate, quantitative manner) to such a decomposition, and

3. differential calculus in open subsets of $\mathbb{R}^n$ along with other specific structural properties of the Euclidean space.

Extension theorems are useful for a tantalizing array of purposes. On the theoretical side, such results constitute a powerful tool in the areas of harmonic analysis and partial differential equations, (cf, e.g., the discussion in [87], [62], [38]) while on the practical side they have found to be useful in applied math (cf. [13],[84] for applications to image processing). Depending on the specific goals one has in mind, the extension problem with preservation of smoothness may acquire various nuances in its formulation. For example, one aspect which makes the extension problem for Lipschitz functions more delicate is the issue of preservation of the Lipschitz constant for vector-valued functions, in which scenario, a different technology than what is developed in [68],[91],[58] has to be employed[1]. Some of the early references dealing with this issue are due to F.A. Valentine in [88] and [89].

The work initiated in McShane and Whitney in the 1930's continues to exert a significant degree

[1]The technology formulated in [68], [91],and [58] imply the existence of $\mathbb{R}^n$-valued extension of a vector-valued Lipschitz function but with Lipschitz constant less than or equal to a dimensional constant multiple of the original.

of influence, and the extension problem continues to be an active area of research. For example, the monograph [46] by A. Jonsson and H. Wallin is devoted to establishing Whitney-type extension results for (arrays of) functions defined on closed subsets of $\mathbb{R}^n$ whose smoothness is measured on Besov and Triebel-Lizorkin scales (intrinsically defined on those closed sets). Also, in [11], Y. Brudnyi and P. Shvartsman have produced intrinsic characterizations of the restrictions to a given closed subset of $\mathbb{R}^n$ of functions from $\mathscr{C}^{1,\omega}(\mathbb{R}^n)$ and, building on this work, in a series of papers (cf. [22]-[25] and the references therein) C. Fefferman has further developed this circle of ideas by producing certain sharp versions of Whitney's extension result in the higher order smoothness case. Applications to partial differential equations and harmonic analysis may be found in [38], [62], [37], as well as the references cited there.

In contrast with C. Fefferman's work, which deals with preservation of higher smoothness ($\mathscr{C}^k$ with $k \geq 1$, i.e. functions whose partial derivatives exists and are continuous up to order k) in a Euclidean setting, our goal here is the study of the extension problem for Hölder functions (or more generally $\dot{\mathscr{C}}^\omega$, i.e., functions whose moduli of continuity are β-subadditive cf. (3.2) in Definition 3.1 and (3.12)-(3.13)) but in the much more general geometric setting of quasi-metric spaces.

Our main result falls into two categories addressing two types of extension problems for Hölder (or $\dot{\mathscr{C}}^\omega$) functions. The first is a generalization of McShane's result from metric spaces and Lipschitz functions to general quasi-metric spaces and Hölder functions. As in Mcshane's original work, this is accomplished by means of a nonlinear algorithm. Our second main result is more akin to the work of Whitney in [91] in that we construct a bounded linear operator $\mathscr{E}$ for Hölder (or $\dot{\mathscr{C}}^\omega$) functions in a quasi-metric space which is *geometrically doubling*. The latter, amounts to the condition that any ball may be covered by at most a fixed number of balls twice as small as the original one (cf. Definition 6.1). More specifically, a version of the two referred to above, slightly adjusted to fit the nature of the current discussion is as follows.

Theorem 1.1. *Let (X, ρ) be a quasi-metric space and ω be a modulus of continuity. Fix a nonempty set $E \subseteq X$. Then any function in $\dot{\mathscr{C}}^\omega(E, \rho)$ may be extended with preservation of Hölder property to the entire set X. Moreover, there exist a constant $C > 0$ which depends only on the quasi-distance ρ and modulus of continuity ω such that*

$$\begin{aligned} &\forall\, g \in \dot{\mathscr{C}}^\omega(E, \rho) \quad \exists\, f \in \dot{\mathscr{C}}^\omega(X, \rho) \text{ such that} \\ &g = f|_E \quad \text{and} \quad \|f\|_{\dot{\mathscr{C}}^\omega(X,\rho)} \leq C \|g\|_{\dot{\mathscr{C}}^\omega(E,\rho)}. \end{aligned} \tag{1.4}$$

See Theorem 3.2 in the body of the thesis, and (3.12)- (3.13) for the definitions of $\dot{\mathscr{C}}^\omega(E, \rho)$ and $\|\cdot\|_{\dot{\mathscr{C}}^\omega(X,\rho)}$.

Theorem 1.2. *Let (X, ρ) be a geometrically doubling quasi-metric space, ω be a modulus of continuity and assume that E is a nonempty, closed subset of X. Then there exists a linear operator $\mathscr{E}$, extending real-valued continuous functions defined on E into continuous real-valued functions defined on X with the property*

$$\mathscr{E} : \dot{\mathscr{C}}^\omega(E, \rho) \longrightarrow \dot{\mathscr{C}}^\omega(X, \rho) \tag{1.5}$$

is well-defined and bounded.

Again, the reader is referred to Theorem 7.1 below for a more precise formulation of Theorem 1.2. To the best of our knowledge, this is the first time the extension problem, addressed in Theorems 1.1 and 1.2, has been considered in the setting of quasi-metric spaces. While the strategy for dealing with Theorem 1.1 is related to the work in [68] (which dealt with metric spaces and Lipschitz functions) there are significant differences for the setting of quasi-metric spaces. To make up for the loss of structure when considering quasi-metric spaces, as opposed to metric spaces, we shall employ a sharp metrization theorem (cf. Theorem 2.1) recently established in [70] which, in turn, extends work by R.A. Macías and C. Segovia in [66]-[67]. Among other things, this allows us to identify a range of exponents, β, for which the space of Hölder functions of order β is plentiful and which is much larger than the one considered in [66]-[67].

Our strategy for dealing with Theorem 1.2 makes use of Theorem 1.1. More specifically, we use Theorem 1.1 in order to prove a quantitative Urysohn's Lemma. Recall that, classically Urysohn's lemma asserts that in a locally compact Hausdorff topological space any compact set can be separated from a closed set by a continuous function provided the two sets in question are disjoint. That is, in this scenario there exist a continuous function which is identically one on the compact set and identically zero on the closed set. In contrast with Urysohn's lemma, which is of a purely topological nature, the quantitative aspect of our result alluded to above has to do with the fact that this topological separation property can be done via Hölder (or $\dot{\mathscr{C}}^{\omega}$) functions with control of the Hölder semi-norms in terms of the quasi-distance between the two sets being separated. For more details on this matter the reader is referred to Theorem 4.1 in the body of the thesis.

In turn, this quantitative Urysohn lemma is used to produce a Hölder (or $\dot{\mathscr{C}}^{\omega}$) partition of unity associated with a Whitney decomposition of an open set in a geometrically doubling quasi-metric space which exhibits natural scaling-like properties. Roughly speaking, the above scaling property regards the correlation between the Hölder semi-norms of the bump functions $(\varphi_j)_j$ to the degree of separation between the level sets (i.e. the sets where $\varphi_j \equiv 0$ and $\varphi_j \equiv 1$) in a dilation invariant-like fashion. A precise formulation may be found in Theorem 5.1. The Whitney decomposition mentioned above refers to Theorem 6.1 in the thesis which states that any proper, nonempty, open subset of geometrically doubling quasi-metric space can be written as the union of countably many quasi-metric balls whose radii are proportional to their distance to the complement of the given open set and which, among other things, have finite overlap.

The final step in the proof of Theorem 1.2 is to set up a linear Whitney-type extension operator based upon the Whitney decomposition and Whitney-like partition of unity results just described. More specifically, given a closed subset E of a geometrically doubling quasi-metric space X, the extension operator we consider is

$$(\mathscr{E}f)(x) := \begin{cases} f(x) & \text{if } x \in E, \\ \sum\limits_j f(p_j)\varphi_j(x) & \text{if } x \in X \setminus E, \end{cases} \qquad \forall\, x \in X. \tag{1.6}$$

where f is an arbitrary real-valued function defined on E, $(p_j)_j$ is a suitably chosen sequence of points in E and $(\varphi_j)_j$ is a partition of unity of the type described above, associated to a Whitney decomposition of the open set $X \setminus E$.

Although the proof of Theorem 1.2 retains, in a broad sense, the strategy presented in [87], the execution is necessarily different given the minimality of the structures involved in the setting we are considering here (compare with 1-3 on page 3). While this degree of generality is certainly desirable given the large spectrum of applications of such a result, it is interesting to note that the absence of miracles associated with differentiability, vector space structure, and Euclidean geometry actually better elucidates the nature of the phenomenon at hand.

Chapters 2-7 make up the first portion of my thesis and contain results obtained in collaboration with M. Mitrea in [8]. The layout of the first part of the thesis is as follows. In Chapter 2 we review basic terminology and results pertaining to quasi-metric spaces. In particular, here we record a sharp metrization theorem, recently established in [70] extending earlier work in [66] and [67]. The main result in Chapter 3 is Theorem 1.1, generalizing [68, Theorem 1, p. 838]. This extension result is valid in general quasi-metric spaces although, the underlying extension algorithm is nonlinear. Chapter 4 deals with separation properties for Hölder functions. The main result here is Theorem 4.1 which may be regarded as a quantitative version of the classical Urysohn lemma. This is of independent interest. Having established the quantitative separation results from Chapter 4, in Chapter 5 we prove the existence of a Whitney-type partition of unity consisting of Hölder functions (see Theorem 5.1. Again, this is useful in and of itself. Moving on, in Chapter 6 we present a Whitney-like decomposition result which extends work by R. Coifmann and G. Weiss in [15, Theorem 3.1, p. 71] and [16, Theorem 3.2, p. 623]. See Theorem 6.1. Finally, in Chapter 7, we formulate and prove our main result in this part of the thesis, Theorem 7.1, which is an extension result for Hölder functions in geometrically doubling quasi-metric spaces.

We wish to emphasize that our generalization of McShane and Whitney's results is done under minimal structural assumptions and without compromising the quantitative aspects of the results in question. In addition, a significant number of preliminary results proved in this part of the thesis (e.g. quantitative Urysohn lemma, separation properties of Hölder functions, Whitney partition of unity, and Whitney decomposition) are of independent interest and should be useful for a large variety of applications to problems of analysis on quasi-metric spaces.

Chapters 8-17 constitute the second portion of my thesis. The results contained in these chapters are based on work recently completed in collaboration with D. Brigham, V. Maz'ya, M. Mitrea, and E. Ziadé [9]. The second part of this thesis may be regarded as having two parts which intertwine closely. One part is of a predominantly geometric flavor and is aimed at describing the smoothness of domains (as classically formulated in analytical terms) in a purely geometric language. The other, having a more pronounced analytical nature, studies how the ability of expressing regularity in a geometric fashion is helpful in establishing sharp results in partial differential equations. We begin by motivating the material belonging to the first part just described.

Over the past few decades, analysis on classes of domains defined in terms of specific geometrical and measure theoretical properties has been a driving force behind many notable advances in partial differential equations and harmonic analysis. Examples of categories of domains with analytic and geometric measure theoretic characteristics are specifically designed to meet the demands and needs of work in the aforementioned fields include the class of nontangentially accessible (NTA) domains introduced in [44] by D. Jerison and C. Kenig (NTA domains form the most general class of regions where the pointwise nontangential behavior of harmonic functions at boundary points is meaningful), the class of (ε, δ)-domains considered in [45] by P. Jones (these are the most general type of domains known to date for which linear extension operators which preserve regularity measured on Sobolev scales may be constructed), uniformly rectifiable (UR) domains introduced in [17] by G. David and S. Semmes (making up the largest class of domains with the property that singular integral operators of Calderón-Zygmund type defined on their boundaries are continuous on L^p, $1 < p < \infty$), and the class of Semmes-Kenig-Toro (SKT) domains defined in [41] (SKT domains make up the most general class of domains for which Fredholm theory for boundary layer potentials, as originally envisioned by I. Fredholm, can be carried out).

In the process, more progress has been registered in our understanding of more familiar (and widely used) classes of domains such as the family of Lipschitz domains, as well as domains exhibiting low regularity assumptions. For example, the following theorem, which characterizes the smoothness of a domain of locally finite perimeter in terms of the regularity properties of the geometric measure theoretic outward unit normal, has been recently proved in [40]:

Theorem 1.3. *Assume that Ω is an open, nonempty, proper subset of $\mathbb{R}^n$ which is of locally finite perimeter and which lies on only one side of its topological boundary, i.e.,*

$$\partial\Omega = \partial(\overline{\Omega}). \tag{1.7}$$

Denote by ν the outward unit normal to Ω, defined in the geometric measure theoretic sense at each point belonging to $\partial^*\Omega$, the reduced boundary of Ω. Finally, fix $\alpha \in (0, 1]$. Then Ω is locally of class $\mathscr{C}^{1,\alpha}$ if and only if ν extends to an S^{n-1}-valued function on $\partial\Omega$ which is locally Hölder of order α.

In particular,

$$\begin{gathered}\Omega \text{ is a locally } \mathscr{C}^{1,1}\text{-domain } \Longleftrightarrow \\ \text{the Gauss map } \nu : \partial^*\Omega \to S^{n-1} \text{ is locally Lipschitz.}\end{gathered} \tag{1.8}$$

Finally, corresponding to the limiting case $\alpha = 0$, one has that Ω is a locally $\mathscr{C}^1$ domain if and only if the Gauss map $\nu : \partial^*\Omega \to S^{n-1}$ has a continuous extension to $\partial\Omega$.

Open subsets of $\mathbb{R}^n$ (of locally finite perimeter) whose outward unit normal is Hölder are typically called Lyapunov domains (cf., e.g., [35], [36, Chapter I]). Theorem 1.3 shows that, with this definition, Lyapunov domains are precisely those open sets whose boundaries may be locally described by graphs of functions with Hölder first order derivatives (in a suitable system of coordinates). All these considerations are of an analytical, or measure theoretical flavor.

By way of contrast, in this part of the thesis we are concerned with finding an intrinsic description of a purely geometrical nature for the class of Lyapunov domains in $\mathbb{R}^n$. In order to be able to elaborate, let us define what we term here to be an hour-glass shape. Concretely, given $a, b > 0$ and $\alpha \in [0, +\infty)$, introduce

$$\mathscr{H}\mathscr{G}^{\alpha}_{a,b} := \big\{x \in \mathbb{R}^n : a|x|^{1+\alpha} < |x_n| < b\big\}. \tag{1.9}$$

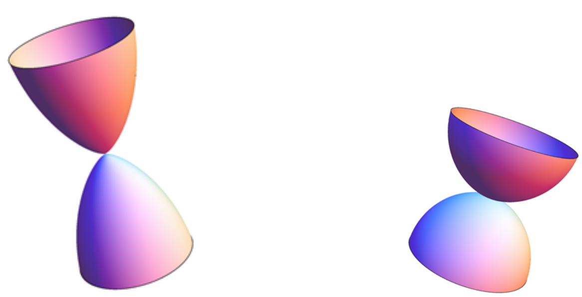


Figure 1. *The figure on the left is an hour-glass shape with α near 0, while the figure depicted on the right is an hour-glass shape with α near 1.*

With this piece of terminology, one of our geometric regularity results may be formulated as follows.

Theorem 1.4. *A nonempty, open set $\Omega \subseteq \mathbb{R}^n$ with compact boundary is Lyapunov if and only if there exist $a, b > 0$ and $\alpha \in (0, 1]$ with the property that for each $x_0 \in \partial\Omega$ there exists an isometry $\mathcal{R} : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\mathcal{R}(0) = x_0 \quad \textit{and} \quad \partial\Omega \cap \mathcal{R}\big(\mathscr{H}\mathscr{G}^{\alpha}_{a,b}\big) = \varnothing. \tag{1.10}$$

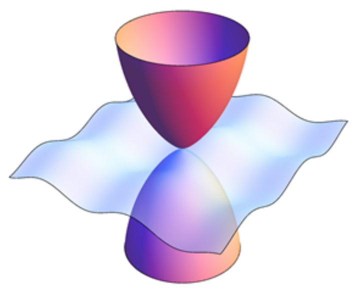


Figure 2. *Threading the boundary of a domain Ω in between the two rounded components of an hour-glass shape with direction vector along the vertical axis.*

The reader is referred to Theorem 12.3 in the body of the thesis for a more precise statement, which is stronger than Theorem 1.4 on two accounts: it is local in nature, and it allows for more general regions than those considered in (1.9). See (1.15) in this regard. Equally important, Theorem 12.3 makes it clear that the Hölder order of the normal is precisely the exponent $\alpha \in (0, 1]$ used in the definition of the hour-glass region (1.9). As a corollary of Theorem 1.4, we note the following purely geometric characterization of domains of class $\mathscr{C}^{1,1}$: *a nonempty, open set $\Omega \subseteq \mathbb{R}^n$ with compact boundary is of class $\mathscr{C}^{1,1}$ if and only if it satisfies a uniform two-sided ball condition.* The latter condition amounts to requesting that there exists $r > 0$ along with a function $h : \partial\Omega \to S^{n-1}$ with the property that

$$B(x + rh(x), r) \subseteq \Omega \quad \text{ and } \quad B(x - rh(x), r) \subseteq \mathbb{R}^n \setminus \Omega \quad \text{ for all } \ x \in \partial\Omega. \tag{1.11}$$

The idea is that the configuration consisting of two open, disjoint, congruent balls in $\mathbb{R}^n$ sharing a common boundary point may be rigidly transported so that it contains an hour-glass region $\mathscr{HG}^{\alpha}_{a,b}$ with $\alpha = 1$ and some suitable choice of the parameters a, b (depending only on the radius r appearing in (1.11)).

The limiting case $\alpha = 0$ of Theorem 1.4 is also true, although the nature of the result changes in a natural fashion. Specifically, if $a \in (0, 1)$ then, corresponding to $\alpha = 0$, the hour-glass region $\mathscr{HG}^{\alpha}_{a,b}$ from (1.9) becomes the two-component, open, circular, upright, truncated cone with vertex at the origin

$$\Gamma_{\theta,b} := \big\{x \in \mathbb{R}^n : \cos(\theta/2)|x| < |x_n| < b\big\}, \tag{1.12}$$

where $\theta := 2 \arccos{(a)} \in (0, \pi)$ is the (total) aperture of the cone. This yields the following characterization of Lipschitzianity: a nonempty, open set $\Omega \subseteq \mathbb{R}^n$ with compact boundary is a Lipschitz domain if and only if there exist $\theta \in (0, \pi)$ and $b > 0$ with the property that for each $x_0 \in \partial\Omega$ there exists an isometry $\mathcal{R} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\mathcal{R}(x_0) = 0 \quad \text{ and } \quad \mathcal{R}(\partial\Omega) \cap \Gamma_{\theta,b} = \varnothing. \tag{1.13}$$

Our characterizations of Lipschitz domains in terms of uniform cone conditions are of independent interest and, in fact, the result just mentioned is the starting point in the proof of Theorem 1.4. Concretely, the strategy for proving the aforementioned geometric characterization of Lyapunov domains in terms of a uniform hour-glass condition with exponent $\alpha \in (0, 1]$ consists of three steps: (1) show that the domain in question is Lipschitz, (2) show that the unit normal satisfies a Hölder condition of order $\alpha/(\alpha + 1)$, and (3) show that the boundary of the original domain may be locally described as a piece of the graph of a function whose first-order derivatives are Hölder of order α.

In fact, we shall prove a more general result than Theorem 1.4 (cf. Theorem 1.5 below), where the (components of the) hour-glass shape (1.9) are replaced by a more general family of subsets of $\mathbb{R}^n$, which we call pseudo-balls (for the justification of this piece of terminology see item (*iii*) in Lemma 8.1). To formally introduce this class of sets, consider

$$\begin{gathered} R \in (0, +\infty) \ \text{ and } \ \omega : [0, R] \to [0, +\infty) \ \text{ a continuous function} \\ \text{with the properties that } \omega(0) = 0 \text{ and } \omega(t) > 0 \ \ \forall\, t \in (0, R]. \end{gathered} \tag{1.14}$$

Then the pseudo-ball with apex at $x_0 \in \mathbb{R}^n$, axis of symmetry along $h \in S^{n-1}$, height $b > 0$, aperture $a > 0$ and shape function ω as in (1.14), is defined as

$$\mathscr{G}^{\omega}_{a,b}(x_0, h) := \{x \in B(x_0, R) : \, a|x - x_0|\, \omega(|x - x_0|) < h \cdot (x - x_0) < b\}\,. \tag{1.15}$$

For certain geometric considerations, it will be convenient to impose the following two additional conditions on the shape function ω:

$$\lim_{\lambda \to 0^+} \Big(\sup_{t \in (0, \min\{R, R/\lambda\}]} \tfrac{\omega(\lambda\, t)}{\omega(t)} \Big) = 0, \quad \text{and } \omega \text{ strictly increasing.} \tag{1.16}$$

Also, in the second half of part II of this thesis, in relation to problems in partial differential equations, we shall work with functions $\widetilde{\omega} : [0, R] \to [0, +\infty)$ satisfying Dini's integrability condition

$$\int_0^R \frac{\widetilde{\omega}(t)}{t}\, dt < +\infty. \tag{1.17}$$

Of significant interest for us in this thesis is the class of functions $\omega_{\alpha,\beta}$, indexed by pairs of numbers $\alpha \in [0, 1]$, $\beta \in \mathbb{R}$, such that $\beta < 0$ if $\alpha = 0$, defined as follows (convening that $\frac{\beta}{0} := +\infty$ for any $\beta \in \mathbb{R}$):

$$\omega_{\alpha,\beta} : \big[0, \min\{e^{\frac{\beta}{\alpha}}, e^{\frac{\beta}{\alpha-1}}\}\big] \to [0, +\infty), \ \ \omega_{\alpha,\beta}(t) := t^{\alpha}(-\ln t)^{\beta} \text{ if } t > 0 \text{ and } \omega_{\alpha,\beta}(0) := 0. \tag{1.18}$$

Corresponding to $\beta = 0$, abbreviate $\omega_\alpha := \omega_{\alpha,0}$. Note $\omega_{\alpha,\beta}$ satisfies all conditions listed in (1.14), (1.16) and (1.17) given $\alpha \in (0, 1]$ and $\beta \in \mathbb{R}$. In addition, we also have that $t \mapsto \omega_{\alpha,\beta}(t)/t$ is decreasing. However, if $\alpha = 0$ then $\omega_{\alpha,\beta}$ satisfies Dini's integrability condition if and only if $\beta < -1$.

If $\alpha \in (0,1]$ and $a, b > 0$ then, corresponding to ω_α as in (1.18), the pseudo-ball

$$\mathscr{G}^{\alpha}_{a,b}(x_0,h) \quad := \quad \mathscr{G}^{\omega_\alpha}_{a,b}(x_0,h) = \big\{x \in B(x_0,1) \subseteq \mathbb{R}^n : \, a|x-x_0|^{1+\alpha} < h\cdot(x-x_0) < b\big\} \tag{1.19}$$

is designed so that the hour-glass region (1.9) consists precisely of the union between $\mathscr{G}^{\alpha}_{a,b}(0,\mathbf{e}_n)$ and $\mathscr{G}^{\alpha}_{a,b}(0,-\mathbf{e}_n)$, where $\mathbf{e}_n$ is the canonical unit vector along the vertical direction in $\mathbb{R}^n = \mathbb{R}^{n-1}\times\mathbb{R}$. The pseudo-balls (1.19) naturally make the transition between cones and genuine balls in $\mathbb{R}^n$ in the sense that, corresponding to $\alpha = 1$, the pseudo-ball $\mathscr{G}^{1}_{a,b}(x_0,h)$ is a solid spherical cap of an ordinary Euclidean ball, whereas corresponding to the limiting case when one formally takes $\alpha = 0$ in (1.19), the pseudo-ball $\mathscr{G}^{0}_{a,b}(x_0,h)$ is a one-component, circular, truncated, open cone (cf. Lemma 8.1 in the body of the thesis for more details).

In order to state the more general version of Theorem 1.4 alluded to above, we need one more definition. Concretely, call an open set $\Omega \subseteq \mathbb{R}^n$ a domain of class $\mathscr{C}^{1,\omega}$ if, near boundary points, its interior may be described (up to an isometric change of variables) in terms of upper-graphs of $\mathscr{C}^1$ functions whose first-order partial derivatives are continuous with modulus ω. Then a version of Theorem 1.4 capable of dealing with the more general type of pseudo-balls introduced in (1.15) reads as follows.

Theorem 1.5. *Let ω be a function as in* (1.14) *and* (1.16). *Then a nonempty, open, proper subset Ω of $\mathbb{R}^n$, with compact boundary is of class $\mathscr{C}^{1,\omega}$ if and only if there exist $a > 0$, $b > 0$ and two functions $h_\pm : \partial\Omega \to S^{n-1}$ with the property that*

$$\mathscr{G}^{\omega}_{a,b}(x,h_+(x)) \subseteq \Omega \quad \textit{and} \quad \mathscr{G}^{\omega}_{a,b}(x,h_-(x)) \subseteq \mathbb{R}^n\setminus\Omega \quad \textit{for each} \;\; x \in \partial\Omega. \tag{1.20}$$

Moreover, in the case when $\Omega \subseteq \mathbb{R}^n$ is known to be of class $\mathscr{C}^{1,\omega}$, one necessarily has $h_- = -h_+$.

This more general version of Theorem 1.4 is justified by the applications to partial differential equations we have in mind. Indeed, as we shall see momentarily, this more general hour-glass shape is important since it permits a desirable degree of flexibility (which happens to be optimal) in constructing certain types of barrier functions, adapted to the operator in question.

More specifically, in the second half of part II of this thesis we deal with maximum principles for second-order, non-divergence form differential operators. Traditionally, the three most basic maximum principles are labeled as weak, boundary point, and strong (cf. the discussion in [30], [82]). Among these, it is the Boundary Point Principle which has the most obvious geometrical character, both in its formulation and proof. For example, M.S. Zaremba [92], E. Hopf [43] and O.A. Oleinik [78] have proved such Boundary Point Principles in domains satisfying an interior ball condition. Our goal here is to prove a sharper version of their results with the interior ball condition replaced by an interior pseudo-ball condition. In fact, it is this goal that has largely motivated the portion of the research in this thesis described earlier.

Being able to use pseudo-balls as a replacement of standard Euclidean balls allows us to relax both the assumptions on the underlying domain, as well as those on the coefficients of the differential operator by considering semi-elliptic operators with singular lower-order terms (drift). Besides its own intrinsic merit, relaxing the regularity assumptions on the coefficients is particularly significant in view of applications to nonlinear partial differential equations.

To state a version of our main result in this regard (cf. Theorem 14.3), we make one definition. Given a real-valued function u of class $\mathscr{C}^2$ in an open subset of $\mathbb{R}^n$, denote by $\nabla^2 u$ the Hessian matrix of u, i.e., $\nabla^2 u := \big(\partial_i \partial_j u\big)_{1\leq i,j\leq n}$. We then have the following Boundary Point Principle, relating the type of degeneracy in the ellipticity, as well as the nature of the singularities in the coefficients of the differential operator, to geometry of the underlying domain.

Theorem 1.6. *Let Ω be a nonempty, proper, open subset of $\mathbb{R}^n$ and assume that $x_0 \in \partial\Omega$ is a point* *with the property that Ω satisfies an interior pseudo-ball condition at x_0. Specifically, assume that*

$$\mathscr{G}^{\omega}_{a,b}(x_0,h) = \{x \in B(x_0,R) :\ a|x-x_0|\,\omega(|x-x_0|) < h\cdot(x-x_0) < b\} \subseteq \Omega, \tag{1.21}$$

for some parameters $a, b, R \in (0,+\infty)$, a direction vector $h \in S^{n-1}$, and a real-valued shape function $\omega \in \mathscr{C}^0([0,R])$, which is positive and non-decreasing on $(0,R]$, and with the property that the mapping $(0,R] \ni t \mapsto \omega(t)/t \in (0,+\infty)$ is non-increasing. Also, consider a non-divergence form, second-order, differential operator L in Ω acting on functions $u \in \mathscr{C}^2(\Omega)$ according to

$$Lu := -\mathrm{Tr}\big(A\,\nabla^2 u\big) + \vec{b}\cdot\nabla u = -\sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j u + \sum_{i=1}^{n} b^i\partial_i u \quad \text{in } \ \Omega, \tag{1.22}$$

whose coefficients $A = \big(a^{ij}\big)_{1\le i,j\le n} : \Omega \to \mathbb{R}^{n\times n}$ and $\vec{b} = (b^i)_{1\le i\le n} : \Omega \to \mathbb{R}^n$ satisfy

$$\inf_{x\in\mathscr{G}^{\omega}_{a,b}(x_0,h)}\ \inf_{\xi\in S^{n-1}} (A(x)\xi)\cdot\xi \ge 0, \quad (A(x)h)\cdot h > 0 \quad \text{for each } \ x \in \mathscr{G}^{\omega}_{a,b}(x_0,h). \tag{1.23}$$

In addition, suppose that there exists a real-valued function $\widetilde{\omega} \in \mathscr{C}^0\big([0,R]\big)$, which is positive on $(0,R]$ and satisfying Dini's integrability condition $\int_0^R t^{-1}\widetilde{\omega}(t)\,dt < +\infty$, with the property that

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{\frac{\omega(|x-x_0|)}{|x-x_0|}\Big(\mathrm{Tr}\,A(x)\Big)}{\frac{\widetilde{\omega}((x-x_0)\cdot h)}{(x-x_0)\cdot h}\Big((A(x)h)\cdot h\Big)} < +\infty, \tag{1.24}$$

and

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{\max\Big\{0\,,\,\vec{b}(x)\cdot h\Big\} + \Big(\sum\limits_{i=1}^{n}\max\{0,-b^i(x)\}\Big)\omega(|x-x_0|)}{\dfrac{\widetilde{\omega}((x-x_0)\cdot h)}{(x-x_0)\cdot h}\Big((A(x)h)\cdot h\Big)} < +\infty. \tag{1.25}$$

Finally, fix a vector $\vec{\ell} \in S^{n-1}$ for which $\vec{\ell}\cdot h > 0$, and suppose that $u \in \mathscr{C}^0(\Omega\cup\{x_0\})\cap\mathscr{C}^2(\Omega)$ is a function satisfying

$$(Lu)(x) \ge 0 \ \ \text{and} \ \ u(x_0) < u(x) \ \ \text{for each} \ \ x \in \Omega. \tag{1.26}$$

Then

$$\liminf_{t\to 0^+} \frac{u(x_0+t\vec{\ell}) - u(x_0)}{t} > 0. \tag{1.27}$$

For example, if $\partial\Omega \in \mathscr{C}^{1,\alpha}$ for some $\alpha \in (0,1)$ and if ν denotes the outward unit normal to $\partial\Omega$, then (1.27) holds provided $\vec{\ell} \cdot \nu(x_0) < 0$ and the coefficients of the semi-elliptic operator L, as in (14.20), satisfy for some $\varepsilon \in (0, \alpha)$

$$(A(x)\nu(x_0)) \cdot \nu(x_0) > 0 \quad \text{for each } x \in \Omega \text{ near } x_0, \text{ and} \tag{1.28}$$

$$\limsup_{\Omega \ni x \to x_0} \frac{|x - x_0|^{\alpha-\varepsilon}\Big(\operatorname{Tr} A(x)\Big) + |x - x_0|^{1-\varepsilon}|\vec{b}(x)|}{(A(x)\nu(x_0)) \cdot \nu(x_0)} < +\infty. \tag{1.29}$$

Also, it can be readily verified that if the coefficients of the operator L are bounded near x_0, then a sufficient condition guaranteeing the validity of (1.24)-(1.25) is the existence of some $c > 0$ such that

$$(A(x)h) \cdot h \geq c\frac{\big((x - x_0) \cdot h\big)\, \omega(|x - x_0|)}{|x - x_0|\widetilde{\omega}((x - x_0) \cdot h)}, \qquad \forall\, x \in \mathscr{G}^{\omega}_{a,b}(x_0, h). \tag{1.30}$$

This should be thought of as an admissible degree of degeneracy in the ellipticity's uniformity of the operator L (a phenomenon concretely illustrated by considering the case when $\omega(t) = t^\alpha$ and $\widetilde{\omega}(t) = t^\beta$ for some $0 < \beta < \alpha < 1$).

It is illuminating to note that the geometry of the pseudo-ball $\mathscr{G}^{\omega}_{a,b}(x_0, h)$ affects (through its direction vector h and shape function ω) the conditions (1.23)-(1.25) imposed on the coefficients of the differential operator L. This is also the case for the proof of Theorem 1.6 in which we employ a barrier function which is suitably adapted both to the nature of the pseudo-ball $\mathscr{G}^{\omega}_{a,b}(x_0, h)$, as well as to the degree of degeneracy of the ellipticity of the operator L (manifested through $\widetilde{\omega}$ and ω). Concretely, this barrier function is defined at each $x \in \mathscr{G}^{\omega}_{a,b}(x_0, h)$ as

$$v(x) := (x - x_0) \cdot h + C_0 \int_0^{(x-x_0)\cdot h} \int_0^{\xi} \frac{\widetilde{\omega}(t)}{t} dt\, d\xi - C_1 \int_0^{|x-x_0|} \int_0^{\xi} \frac{\omega(t)}{t} \Big(\frac{t}{\xi}\Big)^{\gamma-1} dt\, d\xi, \tag{1.31}$$

where $\gamma > 1$ is a fine-tuning parameter, and $C_0, C_1 > 0$ are suitably chosen constants (depending on Ω and L), whose role is to ensure that v satisfies the properties described below. The linear part in the right-hand side of (1.31) is included in order to guarantee that

$$\vec{\ell} \cdot (\nabla v)(x_0) > 0, \tag{1.32}$$

while the constants C_0, C_1 are chosen such that

$$Lv \leq 0 \ \text{ in } \mathscr{G}^{\omega}_{a,b}(x_0,h), \ \text{ and } \exists\, \varepsilon > 0 \text{ so that } \varepsilon v \leq u - u(x_0) \ \text{ on } \partial\mathscr{G}^{\omega}_{a,b}(x_0,h). \tag{1.33}$$

Then (1.27) follows from (1.32)-(1.33) and the Weak Maximum Principle.

Note that the class of second-order, nondivergence form, differential operators considered in Theorem 1.6 is invariant under multiplication by arbitrary positive functions, and that no measurability assumptions are made on the coefficients.

Although a more refined version of Theorem 1.6 is proved later in the thesis (cf. Theorem 14.3), we wish to note here that this result is already quantitatively optimal. To see this, consider the case when $\Omega := \{x \in \mathbb{R}^n_+ : x_n < 1\}$, the point x_0 is the origin in $\mathbb{R}^n$, and

$$L := -\Delta + \frac{\psi(x_n)}{x_n}\frac{\partial}{\partial x_n} \qquad \text{in } \ \Omega, \tag{1.34}$$

where $\psi : (0,1] \to (0,+\infty)$ is a continuous function with the property that

$$\int_0^1 \frac{\psi(t)}{t}\, dt = +\infty. \tag{1.35}$$

Then, if $\vec{\ell} := \mathbf{e}_n := (0,...,0,1) \in \mathbb{R}^n$ and

$$u(x_1,...,x_n) := \int_0^{x_n} \exp\Big\{-\int_\xi^1 \frac{\psi(t)}{t}\, dt\Big\}\, d\xi, \qquad \forall\,(x_1,...,x_n) \in \Omega, \tag{1.36}$$

it follows that $u \in \mathscr{C}^2(\Omega)$, u may be continuously extended at $0 \in \mathbb{R}^n$ by setting $u(0) := 0$, and $u > 0$ in Ω. Furthermore,

$$\frac{\partial u}{\partial x_n} = \exp\Big\{-\int_{x_n}^1 \frac{\psi(t)}{t}\, dt\Big\}, \ \text{ and } \ \frac{\partial^2 u}{\partial x_n^2} = \frac{\psi(x_n)}{x_n}\exp\Big\{-\int_{x_n}^1 \frac{\psi(t)}{t}\, dt\Big\} = \frac{\psi(x_n)}{x_n}\frac{\partial u}{\partial x_n} \ \text{ in } \Omega, \tag{1.37}$$

from which we deduce that $Lu = 0$ in $\mathbb{R}^n_+$, and $(\nabla u)(0) = 0$, thanks to (1.35). Hence (1.27), the conclusion of the Boundary Point Principle formulated in Theorem 1.6, fails in this case. The sole cause of this breakdown is the inability to find a shape function $\widetilde{\omega}$ satisfying Dini's integrability condition and such that (1.25) holds. Indeed, the latter condition reduces, in the current setting, to

$$\begin{gathered}\limsup_{\mathscr{G}^{\omega}_{a,b}(0,\mathbf{e}_n)\ni x\to 0}\Big(\frac{\max\{0,\vec{b}(x)\cdot\mathbf{e}_n\}}{x_n^{-1}\widetilde{\omega}(x_n)}\Big)<+\infty,\qquad\text{where}\\ \vec{b}(x):=\big(0,...,0,\psi(x_n)/x_n\big)\ \text{ for }\ x=(x_1,...,x_n)\in\Omega,\end{gathered}\tag{1.38}$$

which, if true, would force $\widetilde{\omega}(t)\geq c\,\psi(t)$ for all $t>0$ small (for some fixed constant $c>0$). However, in light of (1.35), this would prevent $\widetilde{\omega}$ from satisfying Dini's integrability condition. This proves the optimality of condition (1.25) in Theorem 1.6. A variant of this counterexample also shows the optimality of condition (1.24). Specifically, let Ω, $\vec{\ell}$, x_0, u be as before and, this time, consider

$$L:=-\Big(\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}+\frac{x_n}{\psi(x_n)}\frac{\partial^2}{\partial x_n^2}\Big)+\frac{\partial}{\partial x_n}\qquad\text{in }\ \Omega.\tag{1.39}$$

Obviously, $Lu=0$ in Ω and Ω satisfies an interior pseudo-ball condition at the origin with shape function $\omega(t):=t$. As such, condition (1.24) would entail (for this choice of ω, after some simple algebra), $\widetilde{\omega}(t)\geq c\psi(t)$ for all $t>0$ small. In concert with (1.35) this would, of course, prevents $\widetilde{\omega}$ from satisfying Dini's integrability condition. Other aspects of the sharpness of Theorem 1.6 are discussed later, in Chapter 15.

As a consequence of our Boundary Point Principle, we obtain a Strong Maximum Principle for a class of non-uniformly elliptic operators with singular (and possibly non-measurable) drift terms. More specifically, we have the following theorem.

Theorem 1.7. *Let Ω be a nonempty, connected, open subset of $\mathbb{R}^n$, and suppose that L, written as in* (1.22)*, is a (possibly non-uniformly) elliptic second-order differential operator in non-divergence form (without a zero-order term) in Ω. Also, assume that for each $x_0\in\Omega$ and each $\xi\in S^{n-1}$ there exists a real-valued function $\widetilde{\omega}=\widetilde{\omega}_{x_0,\xi}$ which is continuous on* $[0,1]$*, positive on* $(0,1]$*, satisfies*

$\int_0^1 \frac{\widetilde{\omega}(t)}{t}\,dt < +\infty$, and with the property that

$$\limsup_{\substack{(x-x_0)\cdot\xi>0\\ x\to x_0}} \frac{\Big(\mathrm{Tr}\,A(x)\Big)+\big|\vec{b}(x)\cdot\xi\big|+\big|\vec{b}(x)\big||x-x_0|}{\frac{\widetilde{\omega}((x-x_0)\cdot\xi)}{(x-x_0)\cdot\xi}\Big(\big(A(x)\xi\big)\cdot\xi\Big)} < +\infty. \tag{1.40}$$

Then if $u \in \mathscr{C}^2(\Omega)$ satisfies $(Lu)(x) \geq 0$ for all $x \in \Omega$ and assumes a global minimum value at some point in Ω, it follows that u is constant in Ω.

See Theorem 16.1 for a slightly more refined version, though such a result is already quantitatively sharp. The following example sheds light in this regard. Concretely, in the n-dimensional Euclidean unit ball centered at the origin, consider

$$L := -\tfrac{1}{n+2}\Delta + \vec{b}(x)\cdot\nabla, \quad \text{where} \quad \vec{b}(x) := \begin{cases} |x|^{-2}x & \text{if } x \in B(0,1)\setminus\{0\}, \\ 0 & \text{if } x = 0. \end{cases} \tag{1.41}$$

and the function $u : \overline{B(0,1)} \longrightarrow \mathbb{R}$ given by $u(x) := |x|^4$ for each $x \in \overline{B(0,1)}$.

It follows that

$$u \in \mathscr{C}^2\big(\overline{B(0,1)}\big), \quad (\nabla u)(x) = 4|x|^2 x \ \text{ and } \ (\Delta u)(x) = 4(n+2)|x|^2, \quad \forall\, x \in \overline{B(0,1)}. \tag{1.42}$$

Consequently,

$$(Lu)(x) = 0 \ \text{ for each } \ x \in B(0,1), \quad u \geq 0 \ \text{ in } \ B(0,1), \ u(0) = 0 \ \text{ and } \ u\big|_{\partial B(0,1)} = 1, \tag{1.43}$$

which shows that the Strong Maximum Principle fails in this case. To understand the nature of this failure, observe that given a function $\widetilde{\omega} : (0,1) \to (0,+\infty)$ and a vector $\xi \in S^{n-1}$, condition (1.40) entails

$$\limsup_{\substack{x\to 0\\ x\cdot\xi>0}} \frac{|x|^{-2}\, x\cdot\xi}{\frac{\widetilde{\omega}(x\cdot\xi)}{x\cdot\xi}} < +\infty \tag{1.44}$$

which, when specialized to the case when x approaches 0 along the ray $\{t\xi : t > 0\}$, implies the existence of some constant $c \in (0,+\infty)$ such that $\widetilde{\omega}(t) \geq c$ for all small $t > 0$. Of course, this would prevent $\widetilde{\omega}$ from satisfying Dini's integrability condition.

In the last part of of this chapter we briefly review some of the most common notational conventions used in the sequel. Throughout the thesis, we shall assume that $n \geq 2$ is a fixed integer, $|\cdot|$

stands for the standard Euclidean norm in $\mathbb{R}^n$, and '$\cdot$' denotes the canonical dot product of vectors in $\mathbb{R}^n$. Also, as usual, S^{n-1} is the unit sphere centered at the origin in $\mathbb{R}^n$ and by $B(x,r)$ we denote the open ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$, i.e., $B(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$. Whenever necessary to stress the dependence of a ball on the dimension of the ambient Euclidean space we shall write $B_n(x,r)$ in place of $\{y \in \mathbb{R}^n : |x-y| < r\}$. We let $\{\mathbf{e}_j\}_{1\leq j\leq n}$ denote the canonical orthonormal basis in $\mathbb{R}^n$. In particular, $\mathbf{e}_n = (0,\dots,0,1) \in \mathbb{R}^n$, and we shall use the abbreviation (x', x_n) in place of $(x_1,\dots,x_n) \in \mathbb{R}^n$. By $0'$ we typically denote the origin in $\mathbb{R}^{n-1}$, often regarded as a subspace of $\mathbb{R}^n$ under the canonical identification $\mathbb{R}^{n-1} \equiv \mathbb{R}^{n-1} \times \{0\}$. Next, given $E \subseteq \mathbb{R}^n$, we use E^c, E°, $\overline{E}$ and ∂E to denote, respectively, the complement of E (relative to $\mathbb{R}^n$, i.e., $E^c := \mathbb{R}^n \setminus E$), the interior, the closure and the boundary of E. One other useful piece of terminology is as follows. Let $E \subseteq \mathbb{R}^n$ be a set of cardinality ≥ 2 and assume that $(X, \|\cdot\|)$ is a normed vector space. Then $\mathscr{C}^\alpha(E,X)$ will denote the vector space of functions $f : E \to X$ which are Hölder of order $\alpha > 0$, i.e., for which

$$\|f\|_{\mathscr{C}^\alpha(E,X)} := \sup_{x,y\in E,\, x\neq y} \frac{\|f(x)-f(y)\|}{|x-y|^\alpha} < +\infty. \tag{1.45}$$

As is customary, functions which are Hölder of order $\alpha = 1$ will be referred to as Lipschitz functions. Also, corresponding to the limiting case $\alpha = 0$, we agree that $\mathscr{C}^0$ stands for the class of continuous functions (in the given context).

More generally, given a modulus of continuity ω, a real-valued function f is said to be of class $\mathscr{C}^\omega$ provided there exists $C \in (0,+\infty)$ such that $|f(x)-f(y)| \leq C\,\omega(|x-y|)$ for $|x-y|$ small. Functions of class $\mathscr{C}^{1,\omega}$ are then defined by requiring that their first-order partial derivatives exist and are in $\mathscr{C}^\omega$.

Finally, by $\operatorname{Tr} A$ and $A^\top$ we shall denote, respectively, the trace and transpose of the matrix A.

Chapter 2

Structure of Quasi-Metric Spaces

We start with some preliminary considerations. Given a nonempty set X, call a function $\rho : X \times X \to [0, +\infty)$ a `quasi-distance` provided for every $x, y, z \in X$,

ρ satisfies

$$\rho(x,y) = 0 \Longleftrightarrow x = y, \quad \rho(x,y) = \rho(y,x), \quad \rho(x,y) \leq C \max\{\rho(x,z), \rho(z,y)\}, \tag{2.1}$$

for some finite constant $C \geq 1$.Call two given quasi-distances $\rho_1, \rho_2 : X \times X \longrightarrow [0, +\infty)$ `equivalent`, and write $\rho_1 \approx \rho_2$, if there exist $C', C'' \in (0, +\infty)$ with the property that

$$C'\rho_1 \leq \rho_2 \leq C''\rho_1, \quad \text{on} \quad X \times X. \tag{2.2}$$

For the remainder of our work, if X is a given set of cardinality ≥ 2, we denote by $\mathfrak{Q}(X)$ the collection of all quasi-distances on X. Also, for each $\rho \in \mathfrak{Q}(X)$ define

$$C_\rho := \sup_{\substack{x,y,z \in X \\ \text{not all equal}}} \frac{\rho(x,y)}{\max\{\rho(x,z), \rho(z,y)\}} \tag{2.3}$$

and note that

$$\forall\, \rho \in \mathfrak{Q}(X) \implies C_\rho \in [1, +\infty), \tag{2.4}$$

$$\rho \text{ ultrametric on } X \iff \rho \in \mathfrak{Q}(X) \text{ and } C_\rho = 1, \tag{2.5}$$

$$\forall\, \rho \in \mathfrak{Q}(X), \;\; \forall\, \beta \in (0, +\infty) \implies \rho^\beta \in \mathfrak{Q}(X) \text{ and } C_{\rho^\beta} = (C_\rho)^\beta. \tag{2.6}$$

In light of the natural equivalence relation on $\mathfrak{Q}(X)$, as described in (2.2), we shall refer to each equivalence class $\mathbf{q} \in \mathfrak{Q}(X)/_{\approx}$ as being a `quasi-metric space structure` on X and for each $\rho \in \mathfrak{Q}(X)$, denote by $[\rho] \in \mathfrak{Q}(X)/_{\approx}$ the equivalence class of ρ. By a `quasi-metric space` we shall understand a pair $(X, \mathbf{q})$ where X is a set of cardinality ≥ 2, and $\mathbf{q} \in \mathfrak{Q}(X)/_{\approx}$. If X is a set of cardinality ≥ 2 and $\rho \in \mathfrak{Q}(X)$, we shall frequently use the simpler notation (X, ρ) in place of $(X, [\rho])$, and still refer to (X, ρ) as a quasi-metric space.

Given a quasi-metric space $(X, \mathbf{q})$ and some $\rho \in \mathbf{q}$ we define

$$B_\rho(x, r) := \{y \in X : \rho(x, y) < r\}. \tag{2.7}$$

to be `quasi-metric ball` (with respect to the quasi-distance ρ) centered at $x \in X$ with radius $r > 0$. For brevity, we shall sometimes refer to a quasi-metric ball centered at a point $x \in X$ with radius $r > 0$ simply as a ρ-`ball` if a specific center and radius are unnecessary for the purpose of the discussion. Call $E \subseteq X$ `bounded` if E is contained in a ρ-ball for some (hence all) $\rho \in \mathbf{q}$. In other words, a set $E \subseteq X$ is bounded, relative to the quasi-metric space structure $\mathbf{q}$ on X, if and only if for some (hence all) $\rho \in \mathbf{q}$ we have $\mathrm{diam}_\rho(E) < +\infty$, where

$$\mathrm{diam}_\rho(E) := \sup\big\{\rho(x, y) : \, x, y \in E\big\}. \tag{2.8}$$

Finally, we note that any quasi-metric space $(X, \mathbf{q})$ has a canonical topology, denoted $\tau_{\mathbf{q}}$, which is (unequivocally) defined as the topology τ_ρ naturally induced by the choice of a quasi-distance ρ in $\mathbf{q}$. The latter, characterized as follows

$$\mathcal{O} \in \tau_\rho \Longleftrightarrow \forall\, x \in \mathcal{O}, \ \exists\, r > 0 \text{ such that } B_\rho(x, r) \subseteq \mathcal{O}. \tag{2.9}$$

We now quickly review a theorem that will necessary in proving our results. On the surface the structure of a quasi-metric space can seem somewhat obscure when compared to the rigid and well studied nature of a metric space. However it is known that the topology induced by the given quasi-distance on a quasi-metric space is metrizable. This type of machinery will allow us to move from metric spaces to the more general setting of quasi-metric spaces while retaining a large number of the basic results in analysis. The details of this discussion are presented below in Theorem 2.1 due to D.Mitrea, I.Mitrea, M. Mitrea and S. Monniaux in [70].

Assume that X is an arbitrary, nonempty set. Given an arbitrary function $\rho : X \times X \to [0, +\infty]$ and an arbitrary exponent $\alpha \in (0, +\infty]$ define the function

$$\rho_\alpha : X \times X \longrightarrow [0, +\infty] \tag{2.10}$$

by setting for each $x, y \in X$

$$\begin{aligned}\rho_\alpha(x,y) := \inf \Big\{ \Big(\sum_{i=1}^{N} \rho(\xi_i, \xi_{i+1})^\alpha \Big)^{\frac{1}{\alpha}} :\ & N \in \mathbb{N} \text{ and } \xi_1, \dots, \xi_{N+1} \in X \\ & \text{(not necessarily distinct) such that } \xi_1 = x \text{ and } \xi_{N+1} = y \Big\},\end{aligned} \tag{2.11}$$

whenever $\alpha < +\infty$ and its natural counterpart corresponding to the case when $\alpha = +\infty$, i.e.,

$$\begin{aligned}\rho_\infty(x,y) := \inf \Big\{ \max_{1 \le i \le N} \rho(\xi_i, \xi_{i+1}) :\ & N \in \mathbb{N},\ \xi_1, \dots, \xi_{N+1} \in X \\ & \text{(not necessarily distinct) such that } \xi_1 = x \text{ and } \xi_{N+1} = y \Big\},\end{aligned} \tag{2.12}$$

and note that

$$\forall\, \rho \in \mathfrak{Q}(X),\ \ \forall\, \alpha \in (0, +\infty] \implies \rho_\alpha \in \mathfrak{Q}(X) \ \text{ and } \ \rho_\alpha \le \rho \ \text{ on } X. \tag{2.13}$$

Theorem 2.1. *Let $(X, \mathbf{q})$ be a quasi-metric space. Assume $\rho \in \mathbf{q}$ is fixed and construct ρ_α as in* (2.11)-(2.12) *where $\alpha := [\log_2 C_\rho]^{-1}$. Finally, fix a finite number $\beta \in (0, \alpha]$. Then the function*

$$d_{\rho,\beta} : X \times X \to [0, +\infty], \qquad d_{\rho,\beta}(x,y) := \big[\rho_\alpha(x,y)\big]^\beta, \qquad \forall\, x, y \in X, \tag{2.14}$$

is a distance on X, i.e. for every $x, y, z \in X$, $d_{\rho,\beta}$ satisfies

$$d_{\rho,\beta}(x,y) = 0 \iff x = y \tag{2.15}$$

$$d_{\rho,\beta}(x,y) = d_{\rho,\beta}(y,x) \tag{2.16}$$

$$d_{\rho,\beta}(x,y) \leq d_{\rho,\beta}(x,z) + d_{\rho,\beta}(z,y) \tag{2.17}$$

and which has the property $d_{\rho,\beta}^{1/\beta} \approx \rho$ with

$$C_\rho^{-2}\rho(x,y) \leq \big[d_{\rho,\beta}(x,y)\big]^{1/\beta} \leq \rho(x,y), \quad \forall\, x, y \in X. \tag{2.18}$$

Moreover, ρ_α satisfies the following Hölder-type regularity condition of order β:

$$\big|\rho_\alpha(x,y) - \rho_\alpha(x,z)\big| \leq \tfrac{1}{\beta} \max\big\{\rho_\alpha(x,y)^{1-\beta}, \rho_\alpha(x,z)^{1-\beta}\big\}\big[\rho_\alpha(y,z)\big]^\beta \tag{2.19}$$

whenever $x, y, z \in X$ (with the understanding that when $\beta > 1$ one also imposes the condition that $x \notin \{y, z\}$).

Convention 2.2. *Given a set X and $\rho \in \mathfrak{Q}(X)$, it is agreed that for the remainder of this work $\rho_\#$ stands for ρ_α, as defined in* (2.11)-(2.12) *for the value $\alpha := [\log_2 C_\rho]^{-1}$ with C_ρ as in* (2.3)*.*

As mentioned previously, the fact that the topology induced by the given quasi-distance on a quasi-metric space is metrizable has been known long before Theorem 2.1. As a matter a fact, a version of Theorem 2.1 was formulated in 1979 by R.A. Macías and C. Segovia [66] with exponent $\alpha := [\log_2(c(2c+1))]^{-1}$ where $c \geq 1$ is as in (2.1). Ever since its original inception, this theorem has played a pivotal role in the area of harmonic analysis. Indeed Macías and Segovia's main contribution was to bring to prominence the quantitative aspects of this result in the setting of quasi-metric spaces. In contrast, Theorem 2.1 improves upon this result by finding the sharpest exponent α such that the Hölder regularity condition (2.19) holds. We wish to stress that the actual optimal value of the Hölder regularity exponent α is not an issue of mere curiosity since this number plays a fundamental role in the theory of function spaces which can be developed on spaces of homogeneous type. It was out of this necessity that a new approach and hence Theorem 2.1 needed to be developed.

Chapter 3

Extensions of Hölder Functions on General Quasi-Metric Spaces

We begin this chapter by developing the class of Hölder functions which we consider in our extension results. This class of functions will serve not only as a natural generalization of the Lipschitz functions considered by McShane, Whitney and Kirszbraun but will also serve as the class of functions having the highest degree of regularity given the minimal geometric structure that a quasi-metric space exhibits.

Definition 3.1. *Let* $(X, \mathbf{q})$ *be a quasi-metric space. Assume* $\rho \in \mathbf{q}$ *and* C_ρ *is as in* (2.3)*. Call a function* $\omega : [0, +\infty) \to [0, +\infty)$ *a* `modulus of continuity` *(with respect to the quasi-distance* ρ*) provided*

- ω *is non-decreasing on* $[0, +\infty)$*, and* (3.1)

- *there exists a finite number* $\beta \in (0, (\log_2 C_\rho)^{-1}]$ *with the property that* (3.2)

  $\omega(r) \leq \inf\{\omega(s) + \omega(t) : s, t \geq 0,\ s^\beta + t^\beta = r^\beta\}$ *for all* $r \geq 0$.

As of now, we demand only minimal conditions on the moduli of continuity however, as we progress with the development of our results we will require more properties. For instance, in Theorem 7.1 we will consider a modulus of continuity ω which, in addition to satisfying the conditions in Definition 3.1, is continuously vanishing at zero but is otherwise strictly positive. More specifically, the latter conditions require

$$\cdot \ \ \omega \text{ is continuous at zero,} \tag{3.3}$$

$$\cdot \ \ \omega(0) = 0 \text{ and } \omega(t) > 0 \ \text{ for } \ t > 0. \tag{3.4}$$

Before proceeding, it is worth noting that (3.2) has a useful consequence which is made precise in the following remark.

Remark 3.1. *Let $(X, \mathbf{q})$ be a quasi-metric space and assume $\rho \in \mathbf{q}$. Consider a modulus of continuity ω and let β be as in* (3.2)*. Then for all $c \geq 0$ there exists*

$k \in \mathbb{N}_0 := \{0, 1, 2, 3, ...\}$ such that

$$\omega(ct) \leq 2^k \omega(t), \quad \textit{for all} \ \ t \geq 0. \tag{3.5}$$

In fact, if $c > 0$ we may take $k := \langle \beta \log_2(c) \rangle$ and $k := 0$ if $c = 0$ where, generally speaking,

$$\langle a \rangle := \begin{cases} \inf\{n \in \mathbb{N}_0 : a \leq n\} & \textit{if } a \geq 0, \\ 0 & \textit{if } a < 0, \end{cases} \qquad a \in \mathbb{R}. \tag{3.6}$$

Proof. To see (3.5) fix $c \in [0, +\infty)$ and observe that the case when $c = 0$ follows immediately from (3.4) and taking $k := 0$. If $0 < c \leq 1$ then (3.5) is a direct consequence of (3.1) and (3.6) if $k := \langle \beta \log_2(c) \rangle$.

There remains to consider the situation when $c > 1$. Assuming that this is the case, define $k := \langle \beta \log_2(c) \rangle \in \mathbb{N}_0$ and observe the condition (3.2) implies

$$\omega((s^\beta + t^\beta)^{1/\beta}) \leq \omega(s) + \omega(t), \ \text{ for all } s, t \geq 0 \tag{3.7}$$

where when specializing (3.7) to the case when $s = t$ we obtain

$$\omega(2^{1/\beta} t) \leq 2\omega(t), \quad \text{for all } t \geq 0. \tag{3.8}$$

Furthermore, iterating the inequality in (3.8) yields

$$\omega(2^{n/\beta}t) \leq 2^n \omega(t), \quad \text{for all } t \geq 0,\ n \in \mathbb{N}_0. \tag{3.9}$$

Finally, considering (3.1) in conjunction with (3.9) and the fact $\beta \log_2(c) \leq k$ gives

$$\omega(ct) \leq \omega(2^{k/\beta}t) \leq 2^k \omega(t), \ \text{ for all } t \geq 0. \tag{3.10}$$

which ends the proof of (3.5). □

Occasionally we shall refer to the properties described in (3.5) and (3.2) as the *slow-growth condition* and the *β-subadditivity condition* for ω (respectively).

A prototypical example of a function ω satisfying the properties listed in Definition 3.1 is

$$\omega_{c,\beta} : [0, +\infty) \to [0, +\infty), \quad \omega_{c,\beta}(t) := ct^\beta, \quad \text{for} \quad t \in [0, +\infty), \tag{3.11}$$

where $c, \beta \in (0, +\infty)$ are given. Note that $\omega_{c,\beta}$ also satisfies the conditions listed in (3.3)-(3.4).

Let X be a nonempty set and assume that $\rho \in \mathfrak{Q}(X)$, $E \subseteq X$ has cardinality ≥ 2 and ω a modulus of continuity. Given $f : E \longrightarrow \mathbb{R}$, define its (ω, ρ) - `Hölder semi-norm` (of order ω, relative to the quasi-distance ρ) by setting

$$\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho)} := \sup_{x,y \in E,\, x \neq y} \frac{|f(x) - f(y)|}{\omega(\rho(x,y))}. \tag{3.12}$$

Next, if $\mathbf{q}$ is a quasi-metric space structure on X, we define the homogeneous Hölder space $\dot{\mathscr{C}}^\omega(E, \mathbf{q})$ as

$$\begin{aligned} \dot{\mathscr{C}}^\omega(E, \mathbf{q}) \quad &:= \quad \big\{ f : E \to \mathbb{R} : \|f\|_{\dot{\mathscr{C}}^\omega(E,\rho)} < +\infty \text{ for some } \rho \in \mathbf{q} \big\} \\ &= \quad \big\{ f : E \to \mathbb{R} : \|f\|_{\dot{\mathscr{C}}^\omega(E,\rho)} < +\infty \text{ for every } \rho \in \mathbf{q} \big\}. \end{aligned} \tag{3.13}$$

Given a modulus of continuity ω, it follows that $\{\|\cdot\|_{\dot{\mathscr{C}}^\omega(E,\rho)} : \rho \in \mathbf{q}\}$ is a family of equivalent semi-norms on $\dot{\mathscr{C}}^{\omega}(E,\mathbf{q})$. Additionally, observe that if ω satisfies condition (3.3) then

$$f \in \dot{\mathscr{C}}^{\omega}(E,\mathbf{q}) \Longrightarrow \text{ f is continuous on E.} \tag{3.14}$$

In this same setting, if $c, \beta \in (0,+\infty)$ are fixed and $\omega_{c,\beta}$ is defined as in (3.11) then

$$\dot{\mathscr{C}}^{\beta}(E,\mathbf{q}) := \dot{\mathscr{C}}^{\omega_{c,\beta}}(E,\mathbf{q}) \tag{3.15}$$

is the space of Hölder functions of order β. That is, if $f \in \dot{\mathscr{C}}^{\beta}(E,\mathbf{q})$ then there exist a finite constant $M > 0$ such that

$$|f(x) - f(y)| \leq M\rho(x,y)^{\beta}, \qquad \forall\, x, y \in X. \tag{3.16}$$

Furthermore, the space of Lipschitz functions and the associated Lipschitz semi-norm correspond to specializing (3.15) to the case when $\beta = 1$ and $c \in (0,+\infty)$, i.e., are defined as

$$\mathrm{Lip}\,(E,\mathbf{q}) := \dot{\mathscr{C}}^{\omega_{c,1}}(E,\mathbf{q}), \qquad \|\cdot\|_{\mathrm{Lip}\,(E,\rho)} := \|\cdot\|_{\dot{\mathscr{C}}^{\omega_{c,1}}(E,\rho)}, \quad \forall\, \rho \in \mathbf{q}. \tag{3.17}$$

As far as the above spaces are concerned, if $\rho \in \mathfrak{Q}(X)$ is given then we shall sometimes slightly simplify notation and write $\dot{\mathscr{C}}^{\omega}(E,\rho)$, $\mathrm{Lip}\,(E,\rho)$ in place of $\dot{\mathscr{C}}^{\omega}(E,[\rho])$ and $\mathrm{Lip}\,(E,[\rho])$, respectively. Occasionally, we shall refer to functions in $\mathrm{Lip}\,(E,\rho)$ as being ρ-Lipschitz on E.

Some of the most pioneering work regarding the extension of Lipschitz functions in the setting of metric spaces is due to E.J. Mcshane [68], H. Whitney [91], and M.D. Kirszbraun [58]. Below, the goal is to extend this result of McShane, Whitney, and Kirszbraun to the more general context of Hölder scales of quasi-metric spaces. Before proceeding directly with the formulation of this result we prove the following lemma.

Lemma 3.1. *Let $(X,\mathbf{q})$ be a quasi-metric space. Assume $\rho \in \mathbf{q}$ and ω is a modulus of continuity. Suppose that the set $E \subseteq X$ is fixed. Given a family $\{f_i\}_{i\in I}$ of real-valued functions defined on E*

with the property that

$$M := \sup_{i\in I} \|f_i\|_{\dot{\mathscr{C}}^\omega(E,\rho)} < +\infty, \tag{3.18}$$

consider

$$f^*(x) := \sup_{i\in I} f_i(x), \quad f_*(x) := \inf_{i\in I} f_i(x), \qquad \forall\, x\in E. \tag{3.19}$$

Then the following conclusions hold.

(a) *Either $f^*(x) = +\infty$ for every $x \in E$, or $f^* : E \to \mathbb{R}$ is a well-defined function satisfying $\|f^*\|_{\dot{\mathscr{C}}^\omega(E,\rho)} \leq M$.*

(b) *Either $f_*(x) = -\infty$ for every $x \in E$, or $f_* : E \to \mathbb{R}$ is a well-defined function satisfying $\|f_*\|_{\dot{\mathscr{C}}^\omega(E,\rho)} \leq M$.*

Proof. Consider the conclusion in (a). If f^* is not identically $+\infty$ on E then there exists $x_0 \in E$ such that $\sup_{i\in I} f_i(x_0) < +\infty$. On the other hand, condition (3.18) entails that, for each $i \in I$,

$$f_i(x) \leq f_i(y) + M\omega(\rho(x, y)), \qquad \forall\, x, y \in E. \tag{3.20}$$

Using (3.20) with $y := x_0$ then gives $\sup_{i\in I} f_i(x) \leq \sup_{i\in I} f_i(x_0) + M\omega(\rho(x, x_0))$, where the right hand side is finite for every $x \in E$. Thus, $f^* : E \to \mathbb{R}$ with $f^*(x) := \sup_{i\in I} f_i(x)$ for each $x \in E$ is a well-defined function. Moreover, (3.20) readily gives that $f^*(x) \leq f^*(y) + M\omega(\rho(x, y))$ for all $x, y \in E$ hence, ultimately, $\|f^*\|_{\dot{\mathscr{C}}^\omega(E,\rho)} \leq M$. This finishes the proof of (a).

Finally, (b) follows from (a) used for the family $\{-f_i\}_{i\in I}$. $\square$

We now present the generalization of McShane, Whitney, and Kirszbraun's result mentioned above.

Theorem 3.2. *Let $(X, \mathbf{q})$ be a quasi-metric space. Assume $\rho \in \mathbf{q}$ and ω is a modulus of continuity. Finally, fix a nonempty set $E \subseteq X$. Then any function in $\dot{\mathscr{C}}^\omega(E, \mathbf{q})$ may be extended with* *preservation of Hölder property to the entire set X, i.e., one has*

$$\dot{\mathscr{C}}^{\omega}(E,\mathbf{q}) = \{f|_E : \, f \in \dot{\mathscr{C}}^{\omega}(X,\mathbf{q})\}. \tag{3.21}$$

Furthermore, the Hölder semi-norm may be controlled in the process of extending Hölder functions, in the sense that if $\beta \in (0,+\infty)$ is as in (3.2) and C_ρ is as in (2.3) then

$$\begin{aligned} &\forall\, g \in \dot{\mathscr{C}}^{\omega}(E,\mathbf{q}) \quad \exists\, f \in \dot{\mathscr{C}}^{\omega}(X,\mathbf{q}) \text{ such that} \\ &g = f|_E \quad \text{and} \quad \|f\|_{\dot{\mathscr{C}}^{\omega}(X,\rho)} \leq 2^{\langle 2\beta \log_2 C_\rho \rangle} \|g\|_{\dot{\mathscr{C}}^{\omega}(E,\rho)}. \end{aligned} \tag{3.22}$$

As a corollary, the space $\dot{\mathscr{C}}^{\omega}(X,\mathbf{q})$ separates the points in X. In particular, the space $\dot{\mathscr{C}}^{\omega}(X,\mathbf{q})$ contains non-constant functions.

Proof. Let $\beta \in (0,+\infty)$ be as in (3.2), C_ρ as in (2.3) and consider $g \in \dot{\mathscr{C}}^{\omega}(E,\rho)$. It follows from Theorem 2.1 that there exists $\rho_\# \in \mathbf{q}$ with the property that $(\rho_\#)^\beta$ is a distance on X. We define an explicit formula for the extension of the function g as follows:

$$\begin{aligned} &f(x) := \sup\nolimits_{z\in E} g_z^-(x) \quad \forall\, x \in X, \quad \text{where for each } z \in X \\ &\quad \text{we define } g_z^-(x) := g(z) - K\omega(\rho_\#(x,z)) \quad \forall\, x \in X, \end{aligned} \tag{3.23}$$

for $K := 2^{\langle 2\beta \log_2 C_\rho \rangle} \|g\|_{\dot{\mathscr{C}}^{\omega}(E,\rho)} \in (0,+\infty)$. The significance of this choice of K will become apparent shortly. Note that, by Theorem 2.1, (3.7) and (3.1), we may estimate for each $z \in E$ and $x, y \in X$

$$\begin{aligned} |g_z^-(x) - g_z^-(y)| &= K|\omega(\rho_\#(x,z)) - \omega(\rho_\#(y,z))| \\ &\leq K\omega(\rho_\#(x,y)) \leq K\omega(\rho(x,y)). \end{aligned} \tag{3.24}$$

Where the first inequality in (3.24) follows from the fact that

$$\omega(\rho_\#(x,y)) \leq \omega((\rho_\#(x,z)^\beta + \rho_\#(y,z)^\beta)^{1/\beta}) \leq \omega(\rho_\#(x,z)) + \omega(\rho_\#(y,z)), \tag{3.25}$$

and the second is a consequence of (3.1) and (2.13). Hence, for each $z \in E$, we have $g_z^- \in \dot{\mathscr{C}}^{\omega}(X,\mathbf{q})$ with $\sup_{z\in E} \|g_z^-\|_{\dot{\mathscr{C}}^{\omega}(X,\rho)} \leq K < +\infty$. It follows from this and Lemma 3.1 $\|f\|_{\dot{\mathscr{C}}^{\omega}(X,\rho)} \leq K$. Furthermore, the choice of K ensures that $g_z^- \leq g$ on E, for every $z \in E$. Finally, since $g_z^-(z) = g(z)$ for each $z \in E$ we have $f = g$ on E which completes the proof of (3.21)-(3.22).

To justify the last claim in the statement of the theorem, given two distinct points $x_0, x_1 \in X$, apply (3.22) in the case when $E := \{x_0, x_1\}$ and $f : E \to \mathbb{R}$ is defined as $f(x_0) := 0$, $f(x_1) := 1$. The desired conclusions follow. $\square$

Remark 3.2. *In the setting of Theorem 3.2 one could take as an extension of* $g \in \dot{\mathscr{C}}^{\omega}(E, \mathbf{q})$, *the function* $f(x) := \inf_{z \in E} g_z^+(x)$ *for every* $x \in X$ *where for each* $z \in E$ *we define the function* $g_z^+(x) := g(z) + K\omega(\rho_{\#}(x,z))$ *for every* $x \in X$ *with* $K := 2^{\langle 2\beta \log_2 C_\rho \rangle} \|g\|_{\dot{\mathscr{C}}^{\omega}(E,\rho)}$.

According to the last part in Theorem 3.2, if $(X, \mathbf{q})$ is a quasi-metric space, $\rho \in \mathbf{q}$, and ω is a modulus of continuity, then the Hölder space $\dot{\mathscr{C}}^{\omega}(X, \rho)$ contains plenty of non-constant functions.

The extension result presented in Theorem 1.1 should be compared with Theorem 1.2, stated in Chapter 7. Here we only wish to remark that, while Theorem 7.1 yields a stronger conclusion than Theorem 3.2 (in the sense that in the former theorem we manufacture a linear extension operator, compared to the nonlinear extension procedure the latter theorem), the setting in Theorem 7.1 is more restrictive than in Theorem 3.2 as it presupposes that the quasi-metric space in question is geometrically doubling.

Chapter 4

Separation Properties of Hölder Functions

In this chapter we present the quantitative Urysohn lemma mentioned in Chapter 1. Separation results of this type are particularly useful since they lie at the crossroads of analysis and topology, the latter having a more qualitative flavor whereas the former is quantitative by nature. In particular, having knowledge of a separating function's regularity properties allow for a great deal of analysis to be done even in the most general of environments such as quasi-metric spaces. Here we shall improve upon the separation property described in the last part of Theorem 3.2, by establishing a separation result for Hölder functions in the setting of quasi-metric spaces described below in Theorem 4.1.

Before stating this result, let us recall that, given a quasi-metric space (X, ρ) the ρ-*distance* between two arbitrary, nonempty sets $E, F \subseteq X$ is defined as

$$\mathrm{dist}_\rho(E, F) := \inf\{\rho(x, y) : x \in E,\ y \in F\}. \tag{4.1}$$

Corresponding to the case when $E = \{x\}$ for some $x \in X$, we have $\mathrm{dist}_\rho(x, F) := \mathrm{dist}_\rho(\{x\}, F)$ for $F \subseteq X$. Let us also note here that, as a corollary of Lemma 3.1, if (X, ρ) is a quasi-metric space and if ω is a modulus of continuity then for any functions $f, g \in \dot{\mathscr{C}}^\omega(X, \rho)$ it follows that

$$\max\{f, g\} \in \dot{\mathscr{C}}^\omega(X, \rho), \qquad \min\{f, g\} \in \dot{\mathscr{C}}^\omega(X, \rho), \tag{4.2}$$

and

$$\max\left\{\|\max\{f,g\}\|_{\dot{\mathscr{C}}^\omega(X,\rho)}, \|\min\{f,g\}\|_{\dot{\mathscr{C}}^\omega(X,\rho)}\right\} \leq \max\left\{\|f\|_{\dot{\mathscr{C}}^\omega(X,\rho)}, \|g\|_{\dot{\mathscr{C}}^\omega(X,\rho)}\right\}. \tag{4.3}$$

We now present the quantitative Urysohn's lemma mentioned above.

Theorem 4.1. *Let $(X,\mathbf{q})$ be a quasi-metric space. Assume $\rho \in \mathbf{q}$ and that ω is a modulus of continuity which satisfies* (3.4). *Suppose that $F_0, F_1 \subseteq X$ are two nonempty sets with the property that* $\mathrm{dist}_\rho(F_0,F_1) > 0$. *Then, there exists $\psi \in \dot{\mathscr{C}}^\omega(X,\mathbf{q})$ such that*

$$0 \leq \psi \leq 1, \quad on \quad X, \quad \psi \equiv 0 \quad on \quad F_0, \quad \psi \equiv 1 \quad on \quad F_1, \tag{4.4}$$

and for which

$$\|\psi\|_{\dot{\mathscr{C}}^\omega(X,\rho)} \leq 2^{\langle 2\beta \log_2 C_\rho\rangle}\left[\omega(\mathrm{dist}_\rho(F_0,F_1))\right]^{-1} \tag{4.5}$$

where $\beta \in (0,+\infty)$ is as in (3.2) *and C_ρ is as in* (2.3).

Proof. Let $F_0, F_1 \subseteq X$ be two sets such that $\mathrm{dist}_\rho(F_0,F_1) > 0$ and consider $\varphi : F_0 \cup F_1 \to \mathbb{R}$ given by

$$\varphi(x) := \begin{cases} 0 \text{ if } x \in F_0, \\ 1 \text{ if } x \in F_1, \end{cases} \qquad x \in F_0 \cup F_1. \tag{4.6}$$

Notice that if either $x,y \in F_0$ or $x,y \in F_1$ we have as a result of property (3.4) and hypothesis, $|\varphi(x)-\varphi(y)| = 0 \leq \left[\omega(\mathrm{dist}_\rho(F_0,F_1))\right]^{-1}\omega(\rho(x,y))$. Also, if either $x \in F_0$ and $y \in F_1$ or $x \in F_1$ and $y \in F_0$, then

$$|\varphi(x)-\varphi(y)| = 1 \leq \left[\omega(\mathrm{dist}_\rho(F_0,F_1))\right]^{-1}\omega(\rho(x,y)), \tag{4.7}$$

since ω is nondecreasing and, in the current case, $\rho(x,y) \geq \mathrm{dist}_\rho(F_0,F_1) > 0$. All together these imply

$$\varphi \in \dot{\mathscr{C}}^\omega(F_0 \cup F_1, \rho) \text{ and } \|\varphi\|_{\dot{\mathscr{C}}^\omega(F_0\cup F_1,\rho)} \leq \left[\omega(\mathrm{dist}_\rho(F_0,F_1))\right]^{-1}. \tag{4.8}$$

With this in hand, Theorem 3.2 then ensures the existence of a function $\tilde{\varphi} \in \dot{\mathscr{C}}^{\omega}(X, \rho)$ which extends φ and which has the property that if $\beta \in (0, +\infty)$ is as in (3.2) and C_ρ is as in (2.3) then

$$\|\tilde{\varphi}\|_{\dot{\mathscr{C}}^{\omega}(X,\rho)} \leq 2^{\langle 2\beta \log_2 C_\rho \rangle} \big[\omega(\mathrm{dist}_\rho(F_0, F_1))\big]^{-1}. \tag{4.9}$$

At this stage, consider $\psi : X \to \mathbb{R}$ given by

$$\psi := \min\big\{\max\{\tilde{\varphi}, 0\}, 1\big\}. \tag{4.10}$$

By design, the function ψ satisfies (4.4). Moreover, (4.2)-(4.3) yield $\psi \in \dot{\mathscr{C}}^{\omega}(X, \rho)$ along with $\|\psi\|_{\dot{\mathscr{C}}^{\omega}(X,\rho)} \leq \|\tilde{\varphi}\|_{\dot{\mathscr{C}}^{\omega}(X,\rho)}$. This and (4.9) then prove (4.5), completing the proof of the theorem. $\square$

Chapter 5

Whitney-like Partitions of Unity via Hölder Functions

An important tool in harmonic analysis is the Whitney decomposition of an open, nonempty, proper subset $\mathcal{O}$ of a quasi-metric space (X, ρ) into ρ-balls whose distance to the complement of $\mathcal{O}$ in X is proportional to the radius of the ball in question. Frequently, given such a Whitney decomposition, it is useful to have a partition of unity subordinate to it, which is quantitative in the sense that the size of the functions involved is controlled in terms of the size of their respective supports. Details in the standard setting of $\mathbb{R}^n$ may be found in [87, p. 170].

More recently, such quantitative Whitney partitions of unity have been constructed on general metric spaces (see [59, Lemma 2.4, p.339], [33]), and on quasi-metric spaces, as in [67, Lemma 2.16, p. 278]. Here we wish to improve upon the latter result both by allowing a more general set-theoretic framework and by providing a transparent description of the order of smoothness of the functions involved in such a Whitney-like partition of unity for an arbitrary quasi-metric space. Before proceeding, we wish to note that throughout the remainder of the thesis we will use $\mathbf{1}_E$ to denote the characteristic function of a given set E.

Theorem 5.1. *Let $(X, \mathbf{q})$ be a quasi-metric space. Assume $\rho \in \mathbf{q}$ and that ω is a modulus of continuity which satisfies* (3.4). *In this setting, assume that $\{E_j\}_{j\in I}$, $\{\widetilde{E}_j\}_{j\in I}$ and $\{\widehat{E}_j\}_{j\in I}$ are* *three families of nonempty proper subsets of X satisfying the following properties:*

(a) for each $j \in I$ one has $E_j \subseteq \widetilde{E}_j \subseteq \widehat{E}_j$, $r_j := \mathrm{dist}_\rho(E_j, X \setminus \widetilde{E}_j) > 0$ and

$$\mathrm{dist}_\rho(\widetilde{E}_j, X \setminus \widehat{E}_j) \approx r_j \quad \textit{uniformly for } j \in I; \tag{5.1}$$

(b) one has $r_i \approx r_j$ uniformly for $i, j \in I$ such that $\widehat{E}_i \cap \widehat{E}_j \neq \emptyset$;

(c) there exists $N \in \mathbb{N}$ such that $\sum_{j \in I} \mathbf{1}_{\widehat{E}_j} \leq N$;

(d) one has $\bigcup_{j \in I} E_j = \bigcup_{j \in I} \widehat{E}_j$.

Then there exists a finite constant $C \geq 1$, depending only on ρ, β, N, and the proportionality constants in (a) and (b) above, along with a family of real-valued functions $\{\varphi_j\}_{j \in I}$ defined on X such that the following conditions are valid:

(1) for each $j \in I$ one has

$$\varphi_j \in \dot{\mathscr{C}}^\omega(X, \mathbf{q}) \quad \textit{and} \quad \|\varphi_j\|_{\dot{\mathscr{C}}^\omega(X,\rho)} \leq C\omega(r_j)^{-1}; \tag{5.2}$$

(2) for every $j \in I$ one has

$$0 \leq \varphi_j \leq 1 \ \textit{ on } \ X, \quad \varphi_j \equiv 0 \ \textit{ on } \ X \setminus \widetilde{E}_j, \quad \textit{and} \quad \varphi_j \geq 1/C \ \textit{ on } \ E_j; \tag{5.3}$$

(3) one has $\sum_{j \in I} \varphi_j = \mathbf{1}_{\bigcup_{j \in I} E_j} = \mathbf{1}_{\bigcup_{j \in I} \widetilde{E}_j} = \mathbf{1}_{\bigcup_{j \in I} \widehat{E}_j}$.

Proof. Fix ρ and ω as in the statement of the theorem. Based on Theorem 4.1 and property (a), for each $j \in I$ there exists a function $\psi_j \in \dot{\mathscr{C}}^\omega(X, \mathbf{q})$ such that

$$(i)\ \psi_j \equiv 1 \ \text{ on } \ E_j, \quad (ii)\ \psi_j \equiv 0 \ \text{ on } \ X \setminus \widetilde{E}_j, \quad (iii)\ 0 \leq \psi_j \leq 1 \ \text{ on } \ X, \tag{5.4}$$

and

$$\|\psi_j\|_{\dot{\mathscr{C}}^\omega(X,\rho)} \leq 2^{\langle 2\beta \log_2 C_\rho \rangle}\, \omega(r_j)^{-1}, \tag{5.5}$$

where $\beta \in (0, +\infty)$ is as in (3.2) and C_ρ is as in (2.3).

Consider next the function

$$\Psi : \bigcup_{j \in I} E_j \longrightarrow \mathbb{R}, \quad \Psi := \sum_{j \in I} \psi_j, \quad \text{on} \quad \bigcup_{j \in I} E_j, \tag{5.6}$$

and note that Ψ is well-defined and satisfies

$$1 \leq \Psi \leq N \quad \text{on} \quad \bigcup_{j \in I} E_j. \tag{5.7}$$

Indeed, the fact that Ψ is well-defined follows from (c) and (iii) in (5.4), the first inequality is due to (i) and (iii) in (5.4) and the second inequality above is a consequence of (iii) in (5.4), the fact that $E_j \subseteq \widehat{E}_j$ for each $j \in I$, and statement (c) in the hypotheses. Going further, for each $j \in I$ introduce the function

$$\varphi_j : X \longrightarrow \mathbb{R}, \quad \varphi_j := \begin{cases} \psi_j / \Psi & \text{on } \bigcup_{i \in I} E_i, \\ 0 & \text{on } X \setminus \big(\bigcup_{i \in I} E_i\big). \end{cases} \tag{5.8}$$

By the above discussion, for each $j \in I$ the function φ_j is well-defined and, thanks to (5.8), the first inequality in (5.7) and (ii) in (5.4), satisfies

$$0 \leq \varphi_j \leq 1 \;\text{ on }\; X, \quad \varphi_j \equiv 0 \;\text{ on }\; X \setminus \widetilde{E}_j. \tag{5.9}$$

This proves the first two assertions in (2) in the conclusion of the theorem. Also, employing (5.8), (i) in (5.4), and the second inequality in (5.7), we may conclude that

$$\varphi_j = \psi_j / \Psi = 1/\Psi \geq 1/N \quad \text{on} \quad E_j. \tag{5.10}$$

This finishes the proof of (2) provided one chooses $C \geq N$. Going further, by (c) and the second property in (5.9), the sum $\sum_{j \in I} \varphi_j$ is meaningfully defined in $\mathbb{R}$. In fact, from (5.8) and (5.6), this sum is identically equal to one on $\bigcup_{j \in I} E_j$. Using this analysis and (d) finishes the proof of conclusion (3) from the statement of the theorem.

There remains to prove (1). To this end, as a preliminary step we will show that for each $j \in I$, there holds

$$|\psi_j(x) - \psi_j(y)| \leq 2^{\langle 2\beta \log_2 C_\rho \rangle}\, \omega(r_j)^{-1}\omega(\rho(x,y))\big[\mathbf{1}_{\widehat{E}_j}(x) + \mathbf{1}_{\widehat{E}_j}(y)\big], \quad \forall\, x, y \in X. \tag{5.11}$$

In order to prove (5.11), fix $j \in I$ and, based on $\psi_j \in \dot{\mathscr{C}}^\omega(X,\rho)$ and (5.5), estimate for all $x, y \in X$,

$$|\psi_j(x) - \psi_j(y)| \leq \|\psi_j\|_{\dot{\mathscr{C}}^\omega(X,\rho)}\, \omega(\rho(x,y)) \leq 2^{\langle 2\beta \log_2 C_\rho \rangle}\, \omega(r_j)^{-1}\, \omega(\rho(x,y)). \tag{5.12}$$

By construction, $\psi_j \equiv 0$ on $X \setminus \widetilde{E}_j$ so that if $x, y \in X \setminus \widetilde{E}_j$ then (5.11) is obviously true. In the case when either $x \in \widetilde{E}_j$ or $y \in \widetilde{E}_j$, using the fact that $\widetilde{E}_j \subseteq \widehat{E}_j$ we may write

$$\mathbf{1}_{\widehat{E}_j}(x) + \mathbf{1}_{\widehat{E}_j}(y) \geq 1, \tag{5.13}$$

and thus (5.11) follows from (5.13) and (5.12).

Having disposed of (5.11) we focus on proving (1), i.e., show that for each fixed $j \in I$

$$|\varphi_j(x) - \varphi_j(y)| \leq C\omega(r_j)^{-1}\omega(\rho(x,y)), \quad \forall\, x, y \in X, \tag{5.14}$$

for some finite constant $C > 0$, depending only on ρ, β, N, and the proportionality constants in conditions (a) and (b). Fix $j \in I$ and note that (5.14) is obviously true whenever $x, y \in X \setminus \big(\bigcup_{j\in I} E_j\big)$ as the left-hand side in (5.14) vanishes. Consider next the case when $x, y \in \bigcup_{j \in I} E_j$ in which scenario we compute

$$\begin{aligned} |\varphi_j(x) - \varphi_j(y)| &= \left|\frac{\psi_j(x)}{\Psi(x)} - \frac{\psi_j(y)}{\Psi(y)}\right| = \left|\frac{\psi_j(x)\Psi(y) - \psi_j(y)\Psi(x)}{\Psi(x)\Psi(y)}\right| \\ &\leq |\psi_j(x)\Psi(y) - \psi_j(y)\Psi(x)| \\ &\leq |\psi_j(x) - \psi_j(y)|\Psi(y) + |\Psi(x) - \Psi(y)|\psi_j(y) \\ &\leq N|\psi_j(x) - \psi_j(y)| + |\Psi(x) - \Psi(y)|\mathbf{1}_{\widetilde{E}_j}(y) =: I_1 + I_2. \end{aligned} \tag{5.15}$$

The first inequality above follows from the first inequality in (5.7), the second estimate is a consequence of the triangle inequality, and the third one follows from (5.7) and (*ii*)-(*iii*) in (5.4). Moving on, (5.12) immediately gives

$$I_1 \leq 2^{\langle 2\beta \log_2 C_\rho \rangle}\, N\omega(r_j)^{-1}\omega(\rho(x,y)), \qquad \forall\, x,y \in X. \tag{5.16}$$

As for I_2 we make the claim that there exists a finite constant $C > 0$, depending only on ρ, β, N and the proportionality constants in conditions (a) and (b) from the hypotheses, such that

$$I_2 \leq C\omega(r_j)^{-1}\omega(\rho(x,y)), \qquad \forall\, x,y \in \bigcup_{i\in I} E_i. \tag{5.17}$$

To justify this claim, observe that if $y \in \big(\bigcup_{i\in I} E_i\big) \setminus \widetilde{E}_j$ then $I_2 = 0$, so estimate (5.17) is trivially true. Consider next the case when $y \in \big(\bigcup_{i\in I} E_i\big) \cap \widetilde{E}_j$ and denote by $c > 0$ the lower proportionality constant in (5.1). If $\rho(x,y) \geq cr_j$ then, on the one hand (3.1) and the slow-growth condition for ω described in (3.5) imply the existence of a finite constant $C > 0$ such that $\omega(r_j) \leq C\omega(\rho(x,y))$, while on the other hand $I_2 \leq 2N$ by the second inequality in (5.7). Hence (5.17) holds in this case as well. Suppose now that

$$x \in \bigcup_{i\in I} E_i \quad \text{and} \quad y \in \big(\bigcup_{i\in I} E_i\big) \cap \widetilde{E}_j \quad \text{are such that} \quad \rho(x,y) < cr_j. \tag{5.18}$$

Given that $y \in \widetilde{E}_j$ and since by (5.1) and (5.18) we have

$$\operatorname{dist}_\rho(\widetilde{E}_j, X \setminus \widehat{E}_j) \geq cr_j > \rho(y,x), \tag{5.19}$$

which further entails $x \in \widehat{E}_j$. Based on this, the triangle inequality and (5.11), it follows that

$$\begin{aligned} I_2 &= |\Psi(x) - \Psi(y)|\mathbf{1}_{\widehat{E}_j}(x)\mathbf{1}_{\widehat{E}_j}(y) \leq \sum_{i\in I} |\psi_i(x) - \psi_i(y)|\mathbf{1}_{\widehat{E}_j}(x)\mathbf{1}_{\widehat{E}_j}(y) \\ &\leq 2^{\langle 2\beta \log_2 C_\rho \rangle}\, \omega(\rho(x,y)) \sum_{i\in I} \omega(r_i)^{-1}\big[\mathbf{1}_{\widehat{E}_i}(x) + \mathbf{1}_{\widehat{E}_i}(y)\big]\mathbf{1}_{\widehat{E}_j}(x)\mathbf{1}_{\widehat{E}_j}(y) \\ &\leq 2^{\langle 2\beta \log_2 C_\rho \rangle}\, \omega(\rho(x,y))\big\{I_2' + I_2''\big\}, \quad \text{whenever } x,y \text{ are as in (5.18)}, \end{aligned} \tag{5.20}$$

where

$$I_2' := \sum_{i\in I} \omega(r_i)^{-1} \mathbf{1}_{\widehat{E}_i}(x) \mathbf{1}_{\widehat{E}_j}(x) \quad \text{and} \quad I_2^{''} := \sum_{i\in I} \omega(r_i)^{-1} \mathbf{1}_{\widehat{E}_i}(y) \mathbf{1}_{\widehat{E}_j}(y). \tag{5.21}$$

For each non-zero term in I_2' we necessarily have $x \in \widehat{E}_i \cap \widehat{E}_j$ hence $\widehat{E}_i \cap \widehat{E}_j \neq \emptyset$, which further forces $r_i \approx r_j$, by condition (b) in the hypotheses. Thus, using this, property (c) from the hypotheses, and (3.5),

$$I_2' \leq C\omega(r_j)^{-1} \sum_{i\in I} \mathbf{1}_{\widehat{E}_i}(x) \leq CN\omega(r_j)^{-1}, \tag{5.22}$$

where $C > 0$ is a finite constant which depends only on β and the proportionality constant in (b). Similarly, $I_2^{''} \leq C\omega(r_j)^{-1}$ for some finite constant $C > 0$ depending only on N, β and the proportionality constant in (b). Granted the discussion in the paragraph above (5.18), it follows from this and (5.20) that (5.17) holds as stated.

In summary, this analysis shows that the estimate in (5.14) holds whenever $x, y \in X \setminus \big(\bigcup_{i\in I} E_i\big)$, or $x, y \in \bigcup_{i\in I} E_i$. Therefore, in order to finish the proof of (5.14) it remains to consider the case when

$$x \in \bigcup_{i\in I} E_i \quad \text{and} \quad y \in X \setminus \big(\bigcup_{i\in I} E_i\big), \tag{5.23}$$

or vice-versa. Concretely, assume that (5.23) holds (the other case is treated similarly). Then (5.14) is clear when $x \notin \widetilde{E}_j$ since in such a scenario $\varphi_j(x) = \varphi_j(y) = 0$ by the second property in (5.9) and the second condition in (5.23). Thus matters have been reduced to considering the case when

$$x \in \big(\bigcup_{i\in I} E_i\big) \bigcap \widetilde{E}_j \quad \text{and} \quad y \in X \setminus \big(\bigcup_{i\in I} E_i\big) = X \setminus \big(\bigcup_{i\in I} \widehat{E}_i\big), \tag{5.24}$$

where the equality above is a consequence of condition (d) in the hypotheses. In particular $x \in \widetilde{E}_j$ and $y \in X \setminus \widehat{E}_j$ and, hence, based on (a) we have

$$\rho(x, y) \geq \mathrm{dist}_\rho(\widetilde{E}_j, X \setminus \widehat{E}_j) \geq cr_j \tag{5.25}$$

where, as before, $c > 0$ is the lower proportionality constant in (5.1). In this situation, using the definition of φ_j in (5.8), the first inequality in (5.7), (5.25), and the properties (3.1) and (3.5) of ω we may estimate

$$|\varphi_j(x) - \varphi_j(y)| = \varphi_j(x) = \frac{\psi_j(x)}{\Psi(x)} \le \psi_j(x) \le 1 \le C\omega(r_j)^{-1}\omega(\rho(x,y)), \tag{5.26}$$

where $C > 0$ is a finite constant depending on β and the lower proportionality constant in (5.1). This proves the last case in the analysis of (5.14), finishing the proof of (1) in the conclusion of the theorem. The proof of Theorem 5.1 is now complete. □

There are several important instances when the hypotheses of Theorem 5.1 are satisfied. Yet, perhaps the most basic setting in which families of sets $\{E_j\}_{j\in I}$, $\{\widetilde{E}_j\}_{j\in I}$ and $\{\widehat{E}_j\}_{j\in I}$ satisfying the conditions hypothesized in Theorem 5.1 arise in a natural fashion is in relation to the Whitney decomposition of an open subset of a geometrically doubling quasi-metric space (for more details see Comment 6.2 below).

Chapter 6

Whitney Decomposition in Geometrically Doubling Quasi-Metric Spaces

A statement of the Whitney decomposition theorem, which extends work in [15, Theorem 3.1, p. 71] and [16, Theorem 3.2, p. 623] done in the context of bounded open sets in spaces of homogeneous type, is recorded next. Note that we only assume that (X, ρ) is a geometrically doubling quasi-metric space and, perhaps most importantly, our open set $\mathcal{O}$ is not assumed to be bounded. From the point of view of the strategy of the proof, our approach is entirely self-contained and, as opposed to [16], makes no use of Vitali's covering lemma. This is relevant since the the demand of the boundedness of the open sets for which a Whitney type decomposition is shown to exist in [15, Theorem 3.1, p. 71] and [16, Theorem 3.2, p. 623] is an artifact of the use Vitali's covering lemma (which applies to families of balls of bounded radii). In this regard, our proof is more akin to that in the classical setting of Euclidean spaces from [87, Theorem 1.1, p. 167].

Before proceeding, we wish to formally develop the notion of a geometrically doubling quasi-metric space. It is in this setting we seek to present Theorem 1.2.

Definition 6.1. *A quasi-metric space* $(X, \mathbf{q})$ *is called* `geometrically doubling` *if there exists a quasi-distance* $\rho \in \mathbf{q}$ *for which one can find a number* $N \in \mathbb{N}$*, called the* `geometric doubling` `constant of` $(X, \mathbf{q})$*, with the property that any* ρ*-ball of radius* r *in* X *may be covered by at most* N ρ*-balls in* X *of radii* $r/2$*. Finally, if* X *is an arbitrary, nonempty set and* $\rho \in \mathfrak{Q}(X)$*, call* (X, ρ) *geometrically doubling if* $(X, [\rho])$ *is geometrically doubling.*

Note that if $(X, \mathbf{q})$ is a geometrically doubling quasi-metric space then

$$\begin{aligned}&\forall\, \rho \in \mathbf{q}\ \ \forall\, \theta \in (0,1)\ \ \exists\, N \in \mathbb{N} \text{ such that any } \rho\text{-ball of radius } r\\ &\text{in } X \text{ may be covered by at most } N\ \rho\text{-balls in } X \text{ of radii } \theta r.\end{aligned} \tag{6.1}$$

In particular, this ensures that the last part in Definition 6.1 is meaningful. Another useful consequence of the geometric doubling property for a quasi-metric space $(X, \mathbf{q})$ is as follows

$$\begin{aligned}&\text{if } (X, \mathbf{q}) \text{ is a geometrically doubling quasi-metric space then}\\ &\text{the topological space } (X, \tau_{\mathbf{q}}) \text{ is separable.}\end{aligned} \tag{6.2}$$

We wish to note that for the remainder of the thesis we will use $\#E$ to denote the cardinality of a given set E. We now present the Whitney-like decomposition for geometrically doubling quasi-metric spaces.

Theorem 6.1 (Whitney's decomposition)**.** *Let* (X, ρ) *be a geometrically doubling quasi-metric space. Then for each number* $\lambda \in (1, +\infty)$ *there exist constants* $\Lambda \in (\lambda, +\infty)$ *and* $M \in \mathbb{N}$*, both depending only on* C_ρ, λ *and the geometric doubling constant of* (X, ρ)*, and which have the following significance.*

For each proper, nonempty, open subset $\mathcal{O}$ *of the topological space* (X, τ_ρ) *there exist a sequence of points* $\{x_j\}_{j\in\mathbb{N}}$ *in* $\mathcal{O}$ *along with a family of real numbers* $r_j > 0$*,* $j \in \mathbb{N}$*, for which the following properties are valid:*

(1) $\mathcal{O} = \bigcup_{j\in\mathbb{N}} B_\rho(x_j, r_j)$*;*

(2) $\sum_{j\in\mathbb{N}} \mathbf{1}_{B_\rho(x_j, \lambda r_j)} \le M$ *on* $\mathcal{O}$*. In fact, there exists* $\varepsilon \in (0,1)$*, which depends only on* C_ρ, λ *and the*

geometric doubling constant of (X,ρ), with the property that for any $x_0 \in \mathcal{O}$

$$\#\Big\{j \in \mathbb{N}:\, B_\rho\big(x_0, \varepsilon\, \mathrm{dist}_\rho(x_0, X \setminus \mathcal{O})\big) \cap B_\rho(x_j, \lambda r_j) \neq \emptyset\Big\} \leq M. \tag{6.3}$$

(3) $B_\rho(x_j, \lambda r_j) \subseteq \mathcal{O}$ *and* $B_\rho(x_j, \Lambda r_j) \cap \big[X \setminus \mathcal{O}\big] \neq \emptyset$ *for every* $j \in \mathbb{N}$.

(4) $r_i \approx r_j$ *uniformly for* $i, j \in \mathbb{N}$ *such that* $B_\rho(x_i, \lambda r_i) \cap B_\rho(x_j, \lambda r_j) \neq \emptyset$.

Proof. Set $F := X \setminus \mathcal{O}$, so that F is a nonempty, proper, closed subset of (X, τ_ρ). In a first stage, we shall decompose $\mathcal{O}$ into a family of mutually disjoint dyadic layers $(L_n)_{n\in\mathbb{Z}}$ defined as

$$L_n := \{x \in \mathcal{O}:\, 2^{-n-1} \leq \mathrm{dist}_\rho(x, F) < 2^{-n}\}, \qquad \forall\, n \in \mathbb{Z}. \tag{6.4}$$

Clearly, $L_n \cap L_m = \emptyset$ for any two distinct integers n, m, and we claim that

$$\mathcal{O} = \bigcup_{n\in\mathbb{Z}} L_n. \tag{6.5}$$

Indeed, the right-to-left inclusion is a direct consequence of (6.4). To justify the left-to-right inclusion, pick an arbitrary point $x \in \mathcal{O}$. Since $\mathcal{O}$ is open in (X, τ_ρ), it follows that there exists $r > 0$ with the property that $B_\rho(x, r) \subseteq \mathcal{O}$. In particular, we have $\mathrm{dist}_\rho(x, F) \geq r > 0$. Since $F \neq \emptyset$, this further forces

$$\mathrm{dist}_\rho(x, F) \in (0, +\infty) = \bigcup_{n\in\mathbb{Z}} [2^{-n-1}, 2^{-n}), \tag{6.6}$$

from which the desired conclusion readily follows. This finishes the proof of (6.5).

Moving on, assume next that a parameter $\lambda \in (1, +\infty)$ has been given, and fix some number ε which satisfies

$$0 < \varepsilon < 2^{-1}(\lambda C_\rho^3)^{-1}. \tag{6.7}$$

For each $n \in \mathbb{Z}$, use Zorn's lemma to construct a family of points

$$\{x_j^n\}_{j\in I_n} \subseteq L_n, \tag{6.8}$$

where I_n is a set of indices, such that

$$\rho(x_i^n, x_j^n) > \varepsilon C_\rho 2^{-n-1}, \qquad \forall\, i, j \in I_n \text{ with } i \neq j, \tag{6.9}$$

and which is maximal, with respect to the partial order induced by the inclusion of subsets of L_n, with this property. Observe that since the topological space (X, τ_ρ) is separable (cf. (6.2)), we may assume that I_n is at most countable. We then claim that for each $n \in \mathbb{Z}$ we have:

$$\big\{B_\rho(x_j^n, \varepsilon 2^{-n-1})\big\}_{j \in I_n} \quad \text{are mutually disjoint,} \tag{6.10}$$

$$L_n \subseteq \bigcup_{j \in I_n} B_\rho(x_j^n, \varepsilon C_\rho 2^{-n}). \tag{6.11}$$

To justify these properties, note that if $n \in \mathbb{Z}$ is such that one can find $i, j \in I_n$ with the property that there exists $x \in B_\rho(x_i^n, \varepsilon 2^{-n-1}) \cap B_\rho(x_j^n, \varepsilon 2^{-n-1})$ then

$\rho(x_i^n, x) < \varepsilon 2^{-n-1}$ and $\rho(x_j^n, x) < \varepsilon 2^{-n-1}$ which further imply

$$\rho(x_i^n, x_j^n) \leq C_\rho \max\{\rho(x_i^n, x), \rho(x_j^n, x)\} < \varepsilon C_\rho 2^{-n-1}, \tag{6.12}$$

in contradiction with (6.9). This proves (6.10). As far as (6.11) is concerned, from the maximality of the family $\{x_j^n\}_{j \in I_n}$ in the sense described above it follows that for each $x \in L_n$ there exists $j \in I_n$ with the property that $\rho(x, x_j^n) \leq \varepsilon C_\rho 2^{-n-1}$. Hence $x \in B_\rho(x_j^n, \varepsilon C_\rho 2^{-n})$, proving (6.11).

To proceed, we introduce

$$\widehat{L}_n := \{x \in X : \operatorname{dist}_\rho(x, L_n) < \varepsilon \lambda C_\rho^2 2^{-n}\}, \qquad \forall\, n \in \mathbb{Z}, \tag{6.13}$$

then note that, by (6.8), (6.13) and the fact that $C_\rho \geq 1$, we have

$$\bigcup_{j \in I_n} B_\rho(x_j^n, \varepsilon \lambda C_\rho 2^{-n}) \subseteq \widehat{L}_n, \qquad \forall\, n \in \mathbb{Z}. \tag{6.14}$$

The claim we make at this stage is that

$$\widehat{L}_n \subseteq \{x \in X : C_\rho^{-1} 2^{-n-1} \leq \operatorname{dist}_\rho(x, F) \leq C_\rho 2^{-n}\}, \qquad \forall\, n \in \mathbb{Z}. \tag{6.15}$$

To prove this claim, fix $n \in \mathbb{Z}$, pick an arbitrary point $x_0 \in \widehat{L}_n$ and note that this entails $\mathrm{dist}_\rho(x_0, L_n) < \varepsilon\lambda C_\rho^2 2^{-n}$. From this and (6.4) it follows that there exist $x \in L_n$ and $z \in F$ satisfying

$$2^{-n-1} \leq \rho(x,z) < 2^{-n} \quad \text{and} \quad \rho(x,x_0) < \varepsilon\lambda C_\rho^2 2^{-n}. \tag{6.16}$$

Thanks to (6.7), on the one hand we then have

$$\begin{aligned} \mathrm{dist}_\rho(x_0,F) &\leq \rho(x_0,z) \leq C_\rho \max\{\rho(x_0,x),\rho(x,z)\} \\ &\leq C_\rho \max\{\varepsilon\lambda C_\rho^2 2^{-n}, 2^{-n}\} = C_\rho 2^{-n}, \end{aligned} \tag{6.17}$$

which suits our purposes. On the other hand, for every $w \in F$ we may write

$$\begin{aligned} 2^{-n-1} &\leq \mathrm{dist}_\rho(x,F) \leq \rho(x,w) \leq C_\rho \max\{\rho(x,x_0),\rho(x_0,w)\} \\ &\leq \max\{\varepsilon\lambda C_\rho^3 2^{-n}, C_\rho\rho(x_0,w)\}. \end{aligned} \tag{6.18}$$

In turn, given that $\varepsilon\lambda C_\rho^3 2^{-n} < 2^{-n-1}$, this implies $\rho(x_0,w) \geq C_\rho^{-1}2^{-n-1}$ for all $w \in F$. Thus, ultimately,

$$\mathrm{dist}_\rho(x_0,F) \geq C_\rho^{-1}2^{-n-1}, \tag{6.19}$$

and (6.15) follows from (6.17) and (6.19).

Let us now consider the family of intervals $(J_n)_{n\in\mathbb{Z}}$, where

$$J_n := [C_\rho^{-1}2^{-n-1}, C_\rho 2^{-n}], \qquad \forall\, n \in \mathbb{Z}, \tag{6.20}$$

and note that

$$n,m \in \mathbb{Z} \;\text{ and }\; J_n \cap J_m \neq \emptyset \implies |n-m| \leq 1 + 2\log_2 C_\rho. \tag{6.21}$$

As a consequence,

$$\begin{aligned}&\text{the largest number of intervals in the family } \{J_n : n \in \mathbb{Z}\}\\ &\text{which have a nonempty intersection is } \leq 4(1+\log_2 C_\rho).\end{aligned} \tag{6.22}$$

Going further, for every $x \in \mathcal{O}$ define

$$N(x) := \big\{n \in \mathbb{Z} :\, C_\rho^{-1} 2^{-n-1} \leq \mathrm{dist}_\rho(x, F) \leq C_\rho 2^{-n}\big\} \ \text{ and } \ n(x) := \inf N(x). \tag{6.23}$$

Thus,

$$N(x) = \{n \in \mathbb{Z} :\, \mathrm{dist}_\rho(x, F) \in J_n\}, \qquad \forall\, x \in \mathcal{O}, \tag{6.24}$$

which, when used in concert with (6.22), gives the following estimate for the cardinality of $N(x)$:

$$\#\big(N(x)\big) \leq 4(1+\log_2 C_\rho), \qquad \forall\, x \in \mathcal{O}. \tag{6.25}$$

Together, (6.21), (6.23) and (6.25) imply that for any $x \in \mathcal{O}$ we have

$$0 \leq n - n(x) \leq 1 + 2\log_2 C_\rho \ \text{ for any } n \in N(x). \tag{6.26}$$

Suppose next that an arbitrary point $x_0 \in \mathcal{O}$ has been fixed. We then claim that

$$\begin{aligned}&\text{whenever } n \in \mathbb{Z} \text{ and } j \in I_n \text{ are such that}\\ &B_\rho(x_0, \varepsilon\lambda C_\rho 2^{-n}) \cap B_\rho(x_j^n, \varepsilon\lambda C_\rho 2^{-n}) \neq \emptyset\\ &\text{then } \ B_\rho(x_j^n, \varepsilon 2^{-n-1}) \subseteq B_\rho(x_0, \varepsilon\lambda C_\rho^3 2^{-n(x_0)}).\end{aligned} \tag{6.27}$$

To prove this claim, assume that $n \in \mathbb{Z}$ has the property that there exists $j \in I_n$ for which one can find $y \in X$ such that $y \in B_\rho(x_0, \varepsilon\lambda C_\rho 2^{-n})$ and $y \in B_\rho(x_j^n, \varepsilon\lambda C_\rho 2^{-n})$. Then

$$\rho(x_0, x_j^n) \leq C_\rho \, \max\{\rho(x_0, y), \rho(y, x_j^n)\} < \varepsilon\lambda C_\rho^2 2^{-n}, \tag{6.28}$$

which permits us to conclude that

$$x_0 \in B_\rho(x_j^n, \varepsilon\lambda C_\rho^2 2^{-n}). \tag{6.29}$$

By virtue of (6.14)-(6.15), (6.23) and (6.29) we may, in a first stage, deduce

$$n \in N(x_0). \tag{6.30}$$

In a second stage, we note that if $z \in B_\rho(x_j^n, \varepsilon 2^{-n-1})$ then $\rho(z, x_j^n) < \varepsilon 2^{-n-1}$ and, hence,

$$\begin{aligned}\rho(x_0, z) &\leq C_\rho \max\{\rho(x_0, x_j^n), \rho(x_j^n, z)\} \leq C_\rho \max\{\varepsilon\lambda C_\rho^2 2^{-n}, \varepsilon 2^{-n-1}\} \\ &= \varepsilon\lambda C_\rho^3 2^{-n} \leq \varepsilon\lambda C_\rho^3 2^{-n(x_0)}, \end{aligned} \tag{6.31}$$

where the last inequality is a consequence of (6.30) and (6.23). This finishes the proof of the claim made in (6.27). Let us augment this result by observing that, as seen with the help of (6.30) and (6.26), the ratio of the radii of the two balls in the third line of (6.27) satisfies

$$\frac{\varepsilon 2^{-n-1}}{\varepsilon\lambda C_\rho^3 2^{-n(x_0)}} = 2^{n(x_0)-n-1}(\lambda C_\rho^3)^{-1} \in \big[(4\lambda C_\rho^5)^{-1}\,,\, 2^{-1}(\lambda C_\rho^3)^{-1}\big]. \tag{6.32}$$

At this stage, a combination of (6.10), (6.25), (6.30), (6.32) and (6.1) shows that there exists a constant $M \in \mathbb{N}$, depending only on C_ρ, λ and the geometric doubling constant of (X, ρ), which has the property that for every $x_0 \in \mathcal{O}$ we have

$$\#\Big\{(n, j) : \, n \in \mathbb{Z},\ j \in I_n \text{ and } B_\rho(x_0, \varepsilon\lambda C_\rho 2^{-n}) \cap B_\rho(x_j^n, \varepsilon\lambda C_\rho 2^{-n}) \neq \emptyset\Big\} \leq M. \tag{6.33}$$

Hence, in particular,

$$\sum_{n\in\mathbb{Z} \text{ and } j\in I_n} \mathbf{1}_{B_\rho(x_j^n, \varepsilon\lambda C_\rho 2^{-n})} \leq M \quad \text{on} \quad \mathcal{O}. \tag{6.34}$$

Furthermore, from (6.14) and (6.15) we may also deduce that

$$B_\rho(x_j^n, \varepsilon\lambda C_\rho 2^{-n}) \subseteq \mathcal{O} \qquad \text{whenever } n \in \mathbb{Z} \text{ and } j \in I_n. \tag{6.35}$$

Also, (6.5), (6.11) and (6.35) entail

$$\mathcal{O} = \bigcup_{n\in\mathbb{Z} \text{ and } j\in I_n} B_\rho(x_j^n, \varepsilon C_\rho 2^{-n}). \tag{6.36}$$

Finally, from (6.8) and (6.4) we conclude that

$$2^{-n-1} \leq \operatorname{dist}_\rho(x_j^n, X \setminus \mathcal{O}) < 2^{-n}, \qquad \forall\, n \in \mathbb{Z} \text{ and } \forall\, j \in I_n, \tag{6.37}$$

which further implies the existence of a number $\Lambda \in (\lambda, +\infty)$, depending only on C_ρ, λ and the geometric doubling constant of (X, ρ), with the property that

$$B_\rho(x_j^n, \varepsilon\Lambda C_\rho 2^{-n}) \cap \big[X \setminus \mathcal{O}\big] \neq \emptyset, \qquad \forall\, n \in \mathbb{Z} \ \text{ and } \ \forall\, j \in I_n. \tag{6.38}$$

Thus, properties (1)-(3) in the statement of the theorem are going to be verified if we take $\{B_\rho(x_j, r_j)\}_{j\in\mathbb{N}}$ to be a relabeling of the countable family $\big\{B_\rho(x_j^n, \varepsilon C_\rho 2^{-n})\big\}_{n\in\mathbb{Z},\, j\in I_n}$. Property (4) is also implicit in the above construction. This finishes the proof of Theorem 6.1. $\square$

Comment 6.2. Suppose $\mathcal{O}$ is an proper nonempty subset of a geometrically doubling quasi-metric space (X, ρ) and let $\lambda > 1$. Then Theorem 6.1 ensures the existence of a family $\{B_\rho(x_j, r_j)\}_{j\in\mathbb{N}}$ satisfying properties *(1)-(4)* in the conclusion of Theorem 6.1 for this choice of λ. If $\lambda' > 1$ is fixed with the property that $C_\rho < \lambda'$ and $\lambda' C_\rho < \lambda$ and we take $E_j := B_\rho(x_j, r_j)$, $\widetilde{E}_j := B_\rho(x_j, \lambda' r_j)$ and $\widehat{E}_j := B_\rho(x_j, \lambda r_j)$, for each $j \in \mathbb{N}$, then conditions (a)-(d) in Theorem 5.1 are valid for the families $\{E_j\}_{j\in\mathbb{N}}$, $\{\widetilde{E}_j\}_{j\in\mathbb{N}}$, $\{\widehat{E}_j\}_{j\in\mathbb{N}}$ (with the radii r_j's playing the role of the parameters r_j's from the statement of Theorem 5.1). $\blacksquare$

Chapter 7

Extension of Hölder Functions in Geometrically Doubling Quasi-Metric Spaces

In this chapter we formulate and prove a generalization of the work done by Whitney in [91] by constructing a linear extension operator for Hölder functions in the setting of geometrically doubling quasi-metric spaces. In comparison to the nonlinear extension procedure developed in Theorem 3.2, which is valid in any general quasi-metric space, the extension algorithm presented below in Theorem 7.1 is not only linear but is universally bounded. In order to obtain such a linear extension operator we require the quasi-metric space to be geometrically doubling in the sense of Definition 6.1. The latter condition is essential to the proof of Theorem 7.1 as it relies upon the Whitney decomposition established in Chapter 6.

Theorem 7.1. *Let $(X, \mathbf{q})$ be a geometrically doubling quasi-metric space and assume that E is a nonempty, closed subset of the topological space $(X, \tau_{\mathbf{q}})$, where $\tau_{\mathbf{q}}$ is the topology canonically induced on X by the quasi-metric space structure $\mathbf{q}$. Suppose $\rho \in \mathbf{q}$ and ω is a modulus of continuity satisfying* (3.4)*, then there exists a linear extension operator $\mathscr{E}$ such that*

$$\mathscr{E} : \dot{\mathscr{C}}^{\omega}(E, \rho) \longrightarrow \dot{\mathscr{C}}^{\omega}(X, \rho) \tag{7.1}$$

is well-defined, bounded, and which has the following property. Whenever ω satisfies (3.3) *then $\mathscr{E}$*

extends real-valued continuous functions defined on E into continuous real-valued functions defined on X.

Proof. Assume that $(X, \mathbf{q})$ is a geometric doubling quasi-metric space and fix an arbitrary nonempty, closed subset E of $(X, \tau_{\mathbf{q}})$. Also, suppose that $\rho \in \mathbf{q}$ and ω is a modulus of continuity satisfying (3.1)-(3.3). Finally, let $\beta \in (0, +\infty)$ be as in (3.2), C_ρ as in (2.3). If $E = X$, we simply take $\mathscr{E}$ to be the identity operator, so we assume in what follows that $E \neq X$. Pick a constant $\lambda > C_\rho$ and consider and consider the Whitney decomposition $X \setminus E = \bigcup_{j \in \mathbb{N}} B_\rho(x_j, r_j)$ as in Theorem 6.1. Let $\Lambda \in (\lambda, +\infty)$ and $N \in \mathbb{N}$ be as in the conclusion of Theorem 6.1. Next, select $\lambda' \in (C_\rho, \lambda/C_\rho)$ and define the families $\{\widetilde{E}_j\}_{j \in \mathbb{N}}$, $\{\widehat{E}_j\}_{j \in \mathbb{N}}$ as in Comment 6.2, corresponding to these choices of constants. Then, as already noted, the hypotheses of Theorem 5.1 are satisfied, and we consider a partition of unity $\{\varphi_j\}_{j \in \mathbb{N}}$ satisfying the properties listed in conclusion of Theorem 5.1. Finally, for each $j \in \mathbb{N}$ choose a point $p_j \in E$ with the property that

$$\tfrac{1}{2} \operatorname{dist}_\rho\big(p_j, B_\rho(x_j, \lambda' r_j)\big) \leq \operatorname{dist}_\rho\big(E, B_\rho(x_j, \lambda' r_j)\big) \leq \operatorname{dist}_\rho\big(p_j, B_\rho(x_j, \lambda' r_j)\big). \tag{7.2}$$

Hence, since

$$\operatorname{dist}_\rho\big(E, B_\rho(x_j, \lambda' r_j)\big) \approx r_j, \qquad \text{uniformly in } \ j \in \mathbb{N}. \tag{7.3}$$

it follows from this and (7.2) that

$$\operatorname{dist}_\rho\big(p_j, B_\rho(x_j, \lambda' r_j)\big) \approx r_j, \qquad \text{uniformly in } \ j \in \mathbb{N}. \tag{7.4}$$

Given an arbitrary function $f : E \to \mathbb{R}$, we then proceed to define

$$(\mathscr{E} f)(x) := \begin{cases} f(x) & \text{if } x \in E, \\ \sum\limits_{j \in \mathbb{N}} f(p_j)\varphi_j(x) & \text{if } x \in X \setminus E, \end{cases} \qquad \forall\, x \in X, \tag{7.5}$$

and note that in light of (5.3) and *(2)* in the conclusion of Theorem 6.1 we have,

$\mathscr{E}f : X \to \mathbb{R}$ is well-defined. We propose to show that the operator

$$\mathscr{E} : \dot{\mathscr{C}}^{\omega}(E,\rho) \longrightarrow \dot{\mathscr{C}}^{\omega}(X,\rho) \quad \text{is well-defined, linear and bounded.} \tag{7.6}$$

Hence, the goal is to prove that there exists a finite constant $C \geq 0$ with the property that for any $f \in \dot{\mathscr{C}}^{\omega}(E,\rho)$ there holds

$$|(\mathscr{E}f)(x) - (\mathscr{E}f)(y)| \leq C\|f\|_{\dot{\mathscr{C}}^{\omega}(E,\rho)}\,\omega(\rho(x,y)), \qquad \forall\, x,y \in X. \tag{7.7}$$

Obviously, the estimate in (7.7) holds if $C \geq 1$ whenever $x, y \in E$. Consider next the case when $x \in X \setminus E$ and $y \in E$. As a preliminary matter, we remark that

$$j \in \mathbb{N} \ \text{ and } \ x \in B_\rho(x_j, \lambda' r_j) \implies \rho(x, p_j) \approx r_j, \tag{7.8}$$

with proportionality constant depending only on ρ. To justify the claim in (7.8), note that if $x \in B_\rho(x_j, \lambda r_j)$ for some $j \in \mathbb{N}$ then

$$\rho(x,z) \leq C_\rho \max\{\rho(x,x_j), \rho(x_j,z)\} < \lambda C_\rho\, r_j, \qquad \forall\, z \in B_\rho(x_j, \lambda r_j), \tag{7.9}$$

hence, further, for every $z \in B_\rho(x_j, \lambda r_j)$,

$$\rho(x,p_j) \leq C_\rho \max\{\rho(x,z), \rho(z,p_j)\} < C_\rho \max\{\lambda C_\rho\, r_j\,,\, \rho(z,p_j)\}. \tag{7.10}$$

Taking the infimum over all $z \in B_\rho(x_j, \lambda r_j)$ and keeping in mind (7.4) we therefore arrive at the conclusion that

$$\begin{aligned} \rho(x,p_j) &\leq C_\rho \max\Big\{\lambda C_\rho r_j\,,\, \mathrm{dist}_\rho\big(p_j, B_\rho(x_j,\lambda r_j)\big)\Big\} \\ &\leq C_\rho \max\Big\{\lambda C_\rho r_j\,,\, \mathrm{dist}_\rho\big(p_j, B_\rho(x_j,\lambda' r_j)\big)\Big\} \\ &\leq Cr_j. \end{aligned} \tag{7.11}$$

In summary, this analysis shows that there exists $C = C(\rho) \in (0, +\infty)$ for which

$$j \in \mathbb{N} \ \text{ and } \ x \in B_\rho(x_j, \lambda r_j) \implies \rho(x, p_j) \leq C r_j, \tag{7.12}$$

which is a slightly stronger version than what is really needed in (7.8) (however, this will be useful later on). In the opposite direction, if $x \in B_\rho(x_j, \lambda' r_j)$ for some $j \in \mathbb{N}$ then

$$\rho(x, p_j) \geq \mathrm{dist}_\rho\big(p_j, B_\rho(x_j, \lambda' r_j)\big) \geq c\, r_j, \tag{7.13}$$

by appealing once more to (7.4). Since, as before, $c = c(\rho) \in (0, +\infty)$, this concludes the proof of (7.8). As a consequence, of (7.8) and (7.3) we then obtain

$$\rho(x, p_j) \approx \mathrm{dist}_\rho\big(E, B_\rho(x_j, \lambda' r_j)\big), \quad \text{ uniformly in } j \in \mathbb{N} \text{ and } x \in B_\rho(x_j, \lambda' r_j). \tag{7.14}$$

Going further, whenever $y \in E$ and $x \in B_\rho(x_j, \lambda' r_j)$ for some $j \in \mathbb{N}$, (7.14) allows us to estimate

$$\begin{aligned} \rho(y, p_j) &\leq C_\rho \max\{\rho(y, x), \rho(x, p_j)\} \\ &\leq C_\rho \max\Big\{\rho(y, x)\, , \, \mathrm{dist}_\rho\big(E, B_\rho(x_j, \lambda' r_j)\big)\Big\} \leq C\rho(y, x). \end{aligned} \tag{7.15}$$

Hence, for some finite $C = C(\rho) > 0$, independent of x, y, j, we have

$$y \in E, \ j \in \mathbb{N} \text{ and } x \in B_\rho(x_j, \lambda' r_j) \implies \rho(y, p_j) \leq C\rho(x, y). \tag{7.16}$$

Based on (7.16), (3.5), the fact that $f \in \dot{\mathscr{C}}^\omega(E, \rho)$ and the properties of the functions $\{\varphi_j\}_{j \in \mathbb{N}}$, whenever $x \in X \setminus E$ and $y \in E$ we may therefore estimate

$$\begin{aligned} |(\mathscr{E} f)(y) - (\mathscr{E} f)(x)| &= \Big|f(y) - \sum_{j \in \mathbb{N}} f(p_j)\varphi_j(x)\Big| = \Big|\sum_{j \in \mathbb{N}} (f(y) - f(p_j))\varphi_j(x)\Big| \\ &\leq \sum_{j \in \mathbb{N}} |f(y) - f(p_j)|\, \varphi_j(x) = \sum_{\substack{j \in \mathbb{N} \text{ such that} \\ x \in B_\rho(x_j, \lambda' r_j)}} |f(y) - f(p_j)|\, \varphi_j(x) \\ &\leq C\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho)} \sum_{\substack{j \in \mathbb{N} \text{ such that} \\ x \in B_\rho(x_j, \lambda' r_j)}} \omega(\rho(y, p_j))\varphi_j(x) \\ &\leq C\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho)} \omega(\rho(x, y)), \end{aligned} \tag{7.17}$$

since $0 \le \varphi_j \le 1$ for every $j \in \mathbb{N}$, and since

$$\text{the cardinality of } \{j \in \mathbb{N} : \, x \in B_\rho(x_j, \lambda' r_j)\} \text{ is } \le N. \tag{7.18}$$

Of course, estimate (7.17) suits our purposes. The situation when $y \in X \setminus E$ and $x \in E$ is handled similarly, so there remains to treat the case when $x, y \in X \setminus E$, which we now consider. We shall investigate two separate subcases, starting with:

`Subcase I:` *Assume that the points $x, y \in X \setminus E$ are such that*

$$\rho(x,y) < \varepsilon \operatorname{dist}_\rho(x,E) \quad \textit{where} \quad 0 < \varepsilon < \frac{\lambda}{C_\rho(\Lambda C_\rho + \lambda)}. \tag{7.19}$$

The relevance of the choice made for ε will become more apparent later. For now, we wish to mention that such a choice forces $\varepsilon \in (0, 1/C_\rho)$. To get started in earnest, we make the claim that in the above scenario, we have

$$\operatorname{dist}_\rho(x,E) \le \Big(\frac{C_\rho}{1 - \varepsilon C_\rho}\Big) \operatorname{dist}_\rho(y,E). \tag{7.20}$$

Indeed, for every $z \in E$ we may write

$$\operatorname{dist}_\rho(x,E) \le \rho(x,z) \le C_\rho\big(\rho(x,y) + \rho(y,z)\big) \le C_\rho\big(\varepsilon \operatorname{dist}_\rho(x,E) + \rho(y,z)\big), \tag{7.21}$$

hence $(1 - \varepsilon C_\rho) \operatorname{dist}_\rho(x,E) \le C_\rho \, \rho(y,z)$. Taking the infimum over all $z \in E$, (7.20) follows. Moving on, observe that for every $z \in E$ we have

$$(\mathscr{E} f)(x) - (\mathscr{E} f)(y) = \sum_{j \in \mathbb{N}} (f(p_j) - f(z))(\varphi_j(x) - \varphi_j(y)). \tag{7.22}$$

Let us also point out that if $z \in E$ is such that

$$\tfrac{1}{2}\, \rho(x,z) \le \operatorname{dist}_\rho(x,E) \le \rho(x,z) \tag{7.23}$$

then $\rho(x,z) \approx \operatorname{dist}_\rho(x,E) \le \rho(x,p_j)$. In concert with (7.12), this implies

$$j \in \mathbb{N} \text{ and } x \in B_\rho(x_j, \lambda r_j) \implies \rho(p_j, z) \le C_\rho \max\{\rho(p_j,x), \rho(x,z)\} \le C r_j. \tag{7.24}$$

Having established (7.24), we next write formula (7.22) with $z \in E$ as in (7.23) and make use of fact that $f \in \dot{\mathscr{C}}^{\omega}(E, \rho)$ along with (3.5) and the properties of $\{\varphi_j\}_{j \in \mathbb{N}}$ in order to estimate

$$\begin{aligned}
|(\mathscr{E} f)(x) - (\mathscr{E} f)(y)| &\leq \sum_{j \in \mathbb{N}} |f(p_j) - f(z)||\varphi_j(x) - \varphi_j(y)| \\
&\leq C\|f\|_{\dot{\mathscr{C}}^{\omega}(E,\rho)}\, \omega(\rho(x,y)) \sum_{j \in \mathbb{N}} \omega(\rho(p_j, z))\|\varphi_j\|_{\dot{\mathscr{C}}^{\omega}(X,\rho)} \big[\mathbf{1}_{B_\rho(x_j, \lambda' r_j)}(x) + \mathbf{1}_{B_\rho(x_j, \lambda' r_j)}(y)\big] \\
&\leq C\|f\|_{\dot{\mathscr{C}}^{\omega}(E,\rho)}\, \omega(\rho(x,y)) \big(A_x + A_y\big),
\end{aligned} \tag{7.25}$$

where

$$A_x := \sum_{\substack{j \in \mathbb{N} \text{ such that} \\ x \in B_\rho(x_j, \lambda' r_j)}} \omega(\rho(p_j, z))\, \omega(r_j)^{-1} \quad \text{and} \quad A_y := \sum_{\substack{j \in \mathbb{N} \text{ such that} \\ y \in B_\rho(x_j, \lambda' r_j)}} \omega(\rho(p_j, z))\, \omega(r_j)^{-1}. \tag{7.26}$$

Now, (3.5), (7.18) and (7.24) give that $A_x \leq C$, for some finite constant $C = C(\rho, \beta) \geq 0$. In order to derive a similar estimate for A_y, assume that

$$j \in \mathbb{N} \ \text{ is such that } \ y \in B_\rho(x_j, \lambda' r_j). \tag{7.27}$$

Then by (7.20), (7.27), and the fact that $B_\rho(x_j, \Lambda r_j) \cap E \neq \emptyset$ we have

$$\operatorname{dist}_\rho(x, E) \leq \Big(\frac{C_\rho}{1 - \varepsilon C_\rho}\Big) \operatorname{dist}_\rho(y, E) \leq \Lambda \Big(\frac{C_\rho}{1 - \varepsilon C_\rho}\Big) r_j. \tag{7.28}$$

In turn, (7.28) permits us to deduce that

$$\begin{aligned}
\rho(x, x_j) \quad &\leq \quad C_\rho \max\{\rho(x,y), \rho(y, x_j)\} \leq C_\rho \max\Big\{\varepsilon \operatorname{dist}_\rho(x, E)\, ,\, \lambda' r_j\Big\} \\
&\leq \quad C_\rho\, r_j \max\Big\{\varepsilon\, \Lambda \Big(\tfrac{C_\rho}{1 - \varepsilon C_\rho}\Big)\, ,\, \lambda'\Big\} \, < \, \lambda r_j,
\end{aligned} \tag{7.29}$$

where the last inequality is a consequence of the fact that $\lambda' C_\rho < \lambda$ and the way ε has been chosen in (7.19). Estimate (7.29) shows that

$$\text{if } j \text{ is as in (7.27) then } x \in B_\rho(x_j, \lambda r_j). \tag{7.30}$$

With (7.30) in hand, a reference to (7.24) then gives

$$\text{if } j \text{ is as in (7.27) then } \rho(z,p_j) \leq C r_j \text{ whenever } z \text{ is as in (7.23)}, \tag{7.31}$$

for some finite $C = C(\rho) \geq 0$. Having proved (7.31) then the estimate $A_y \leq C$ for some finite constant C depending on ρ and β follows as in the case of A_x, already treated. Altogether, this proves that $A_x + A_y \leq C = C(\rho,\beta) < +\infty$ which, in combination with (7.25), yields the estimate $|(\mathscr{E}f)(x) - (\mathscr{E}f)(y)| \leq C\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho)}\,\omega(\rho(x,y))$, under the hypotheses specified in Subcase I. This bound if of the right order, and this completes the treatment of Subcase I.

`Subcase II:` *With the parameter $\varepsilon > 0$ as in Subcase I, assume that $x, y \in X \setminus E$ are such that $\rho(x,y) \geq \varepsilon\,\mathrm{dist}_\rho(x,E)$.* Consider a point $z \in E$ as in (7.23) and note that, in the current situation, this forces $\rho(x,z) \leq 2\,\mathrm{dist}_\rho(x,E) \leq 2\varepsilon^{-1}\rho(x,y)$. Hence, additionally we have $\rho(y,z) \leq C_\rho \max\{\rho(x,y), \rho(x,z)\} \leq C\rho(x,y)$. Consequently,

$$\begin{aligned} |(\mathscr{E}f)(x) - (\mathscr{E}f)(y)| &\leq |(\mathscr{E}f)(x) - (\mathscr{E}f)(z)| + |(\mathscr{E}f)(z) - (\mathscr{E}f)(y)| \\ &\leq C\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho)}\,\omega(\rho(x,z)) + C\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho)}\,\omega(\rho(z,y)) \\ &\leq C\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho)}\,\omega(\rho(x,y)), \end{aligned} \tag{7.32}$$

by what we have established in the first of the proof (i.e., using (7.17) and (3.5) twice, once for $x \in X \setminus E$ and $z \in E$ and, a second time, for $y \in X \setminus E$ and $z \in E$).

In summary, the above analysis proves that there exists $C = C(\rho,\beta,\varepsilon) \in (0,+\infty)$ with the property that for every $f \in \dot{\mathscr{C}}^\omega(E,\rho)$ we have

$$|(\mathscr{E}f)(x) - (\mathscr{E}f)(y)| \leq C\|f\|_{\dot{\mathscr{C}}^\omega(E,\rho)}\,\omega(\rho(x,y)), \qquad \forall\, x, y \in X. \tag{7.33}$$

This shows that the operator (7.1) is well-defined, linear and bounded.

At this stage, there remains to prove that the operator $\mathscr{E}$ defined in (7.5) has the property that

$$\mathscr{E}f \text{ is continuous on } X \text{ whenever } f : E \to \mathbb{R} \text{ is continuous} \tag{7.34}$$

provided ω satisfies (3.3). To this end, suppose ω satisfies (3.3) and fix an arbitrary continuous function $f : E \to \mathbb{R}$. Note that, by design, $\mathscr{E}f$ is continuous on the open set $X \setminus E$ (since the sum in (7.5) is locally finite and the φ_j's are continuous as a result of (3.14)). There remains to show that $\mathscr{E}f$ is continuous at any point in E. Furthermore, since (as seen from Theorem 2.1) the topology $\tau_{\mathbf{q}}$ is metrizable, it suffices to use the sequential characterization of continuity. Fix $z \in E$ and assume that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of points in X which converges to z in the topology $\tau_{\mathbf{q}}$. Introduce $N_0 := \{n \in \mathbb{N} : x_n \in E\}$ and $N_1 := \{n \in \mathbb{N} : x_n \in X \setminus E\}$. Then, on the one hand,

$$\lim_{N_0 \ni n \to \infty} (\mathscr{E}f)(x_n) = \lim_{N_0 \ni n \to \infty} f(x_n) = f(z), \tag{7.35}$$

since f is continuous on E. On the other hand, for each $n \in N_1$, much as in (7.17) we may estimate

$$|(\mathscr{E}f)(x_n) - (\mathscr{E}f)(z)| \leq \sum_{\substack{j\in\mathbb{N} \text{ such that} \\ x_n \in B_\rho(x_j, \lambda' r_j)}} |f(z) - f(p_j)| \tag{7.36}$$

Let us also note that the version of (7.16) in the notation currently employed reads

$$j \in \mathbb{N} \text{ and } x_n \in B_\rho(x_j, \lambda' r_j) \implies \rho(z, p_j) \leq C\rho(x_n, z), \tag{7.37}$$

for some finite $C = C(\rho) > 0$, independent of $n \in N_1$. Fix an arbitrary $\varepsilon > 0$ and, based on the continuity of f at z, pick $\delta > 0$ with the property that

$$|f(z) - f(w)| < \varepsilon \quad \text{whenever} \quad w \in E \quad \text{is such that} \quad \rho(z, w) < \delta. \tag{7.38}$$

Since $\lim\limits_{N_1 \ni n \to \infty} x_n = z$, it follows that there exists $m \in \mathbb{N}$ with the property that

$$\rho(x_n, z) < \delta/C \text{ for each } n \in N_1 \text{ with the property that } n \geq m, \tag{7.39}$$

where the constant C is as in (7.37). Thus,

$$|(\mathscr{E}f)(x_n) - (\mathscr{E}f)(z)| \leq N\varepsilon, \qquad \text{for every } \ n \in N_1 \ \text{ with } \ n \geq m, \tag{7.40}$$

by (7.36), (7.37), (7.38), and (7.18). Since $\varepsilon > 0$ was arbitrary, it follows from (7.35) and (7.40) that $\mathscr{E}f$ is continuous at z. This completes the justification of (7.34), and finishes the proof of Theorem 7.1. □

Chapter 8

The Geometry of Pseudo-Balls

In this chapter we introduce a category of sets which contains both the cones and balls in $\mathbb{R}^n$, and which we shall call pseudo-balls. This concept is going to play a basic role for the entire subsequent discussion. As a preamble, we describe the class of cones in the Euclidean space. Concretely, by an open, truncated, one-component circular cone in $\mathbb{R}^n$ we understand any set of the form

$$\Gamma_{\theta,b}(x_0,h) := \{x \in \mathbb{R}^n : \cos(\theta/2)\,|x-x_0| < (x-x_0)\cdot h < b\}, \tag{8.1}$$

where $x_0 \in \mathbb{R}^n$ is the vertex of the cone, $h \in S^{n-1}$ is the direction of the axis, $\theta \in (0,\pi)$ is the (full) aperture of the cone, and $b \in (0,+\infty)$ is the height of the cone.

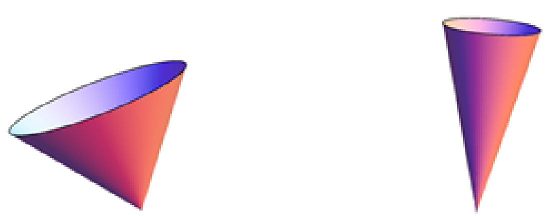


Figure 3. *One-component circular cones. The aperture of the cone on the left is larger than that of the cone on the right.*

Definition 8.1. *Assume* (1.14) *and suppose that the point* $x_0 \in \mathbb{R}^n$, *vector* $h \in S^{n-1}$ *and numbers* $a, b \in (0,+\infty)$ *are given. Then the* $\texttt{pseudo}-\texttt{ball}$ *with apex at* x_0, *axis of symmetry along* h, *height* *b, amplitude a, and shape function ω is defined by*

$$\mathscr{G}^{\omega}_{a,b}(x_0,h) := \{x \in B(x_0,R) \subseteq \mathbb{R}^n : a|x-x_0|\,\omega(|x-x_0|) < h\cdot(x-x_0) < b\}. \tag{8.2}$$

Collectively, a,b and ω constitute the geometrical characteristics of the named pseudo-ball.

In the sequel, given a,b and α positive numbers, abbreviate $\mathscr{G}^{\alpha}_{a,b}(x_0,h) := \mathscr{G}^{\omega_\alpha}_{a,b}(x_0,h)$ with ω_α as in (1.18)*, i.e., define*

$$\mathscr{G}^{\alpha}_{a,b}(x_0,h) := \big\{x \in B(x_0,1) \subseteq \mathbb{R}^n : a|x-x_0|^{1+\alpha} < h\cdot(x-x_0) < b\big\}. \tag{8.3}$$

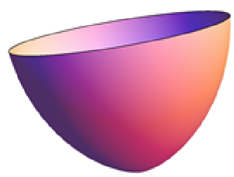


Figure 4. *A pseudo-ball with shape function $\omega(t) = t^{1/2}$.*

Some basic, elementary properties of pseudo-balls are collected in the lemma below. In particular, item (iii) justifies the terminology employed in Definition 8.1.

Lemma 8.1. *Assume* (1.14) *and, in addition, suppose that ω is strictly increasing. Also, fix two parameters $a,b \in (0,+\infty)$, a point $x_0 \in \mathbb{R}^n$ and a vector $h \in S^{n-1}$. Then the following hold.*

(i) The pseudo-ball $\mathscr{G}^{\omega}_{a,b}(x_0,h)$ is a nonempty open subset of $\mathbb{R}^n$ (in fact, it contains a line segment of the form $\{x_0+th : 0<t<\varepsilon\}$ for some small $\varepsilon>0$), which is included in the ball $B(x_0,R)$, and with the property that $x_0 \in \partial\mathscr{G}^{\omega}_{a,b}(x_0,h)$. Corresponding to the choice $x_0 := 0 \in \mathbb{R}^n$ and $h := \mathbf{e}_n \in S^{n-1}$, one has

$$\mathscr{G}^{\omega}_{a,b}(0,\mathbf{e}_n) = \{x = (x',x_n) \in \mathbb{R}^{n-1}\times\mathbb{R} = \mathbb{R}^n : |x| < R,\ a|x|\,\omega(|x|) < x_n < b\}. \tag{8.4}$$

Furthermore,

$$\begin{aligned}&\text{if } b \in (0, R\,\omega(R)) \text{ and if } t_b \in (0,R) \text{ satisfies}\\ &t_b\,\omega(t_b) = b \text{ then } \mathscr{G}^{\omega}_{a,b}(x_0,h) \subseteq B(x_0,t_b).\end{aligned} \tag{8.5}$$

(ii) Assume that $a \in (0,1)$. Then, corresponding to the limiting case when $\alpha = 0$, the pseudo-ball introduced in (8.3) coincides with the one-component, circular, open cone with vertex at x_0, unit axis h, aperture $\theta := 2\arccos a \in (0,\pi)$, and which is truncated at height b. That is,

$$\mathscr{G}^{0}_{a,b}(x_0,h) = \Gamma_{\theta,b}(x_0,h), \qquad \text{for } \ \theta := 2\arccos a \in (0,\pi). \tag{8.6}$$

(iii) In the case when $\alpha = 1$, then for each $a > 0$ the pseudo-ball defined in (8.3) coincides with the solid spherical cap obtained by intersecting the open ball in $\mathbb{R}^n$ with center at $x_0 + h/(2a)$ and radius $r := 1/(2a)$ with the half-space $H(x_0,h,b) := \{x \in \mathbb{R}^n : (x - x_0)\cdot h < b\}$. In other words[1]*,*

$$\mathscr{G}^{1}_{a,b}(x_0,h) = B\big(x_0 + h/(2a), 1/(2a)\big) \cap H(x_0,h,b). \tag{8.7}$$

Furthermore, when and $b \geq 1/a$, one actually has

$$\mathscr{G}^{1}_{a,b}(x_0,h) = B\big(x_0 + h/(2a), 1/(2a)\big).$$

(iv) Let $\mathcal{R} : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry; hence, $\mathcal{R} = T \circ \mathscr{R}$, where $\mathscr{R}$ is a rotation about the origin in $\mathbb{R}^n$ and T is a translation in $\mathbb{R}^n$. Then

$$\mathcal{R}\big(\mathscr{G}^{\omega}_{a,b}(x_0,h)\big) = \mathscr{G}^{\omega}_{a,b}(\mathcal{R}(x_0), \mathscr{R}h). \tag{8.8}$$

In particular, $x_1 + \mathscr{G}^{\omega}_{a,b}(x_0,h) = \mathscr{G}^{\omega}_{a,b}(x_0 + x_1, h)$ for every $x_1 \in \mathbb{R}^n$.

[1] The term "pseudo-ball" has been chosen, *faute de mieux*, primarily because of this observation.

(v) Pick $t_* \in (0,R)$ with the property that $\omega(t_*) < 1$. Then whenever the number b_0 and the angle θ satisfy

$$0 < b_0 < \min\{b, t_*\}, \quad 2\max\Big\{\arccos\big(\omega(t_*)\big)\,,\,\arccos\big(b_0/t_*\big)\Big\} \leq \theta < \pi, \tag{8.9}$$

one has

$$\Gamma_{\theta,b_0}(x_0,h) \subseteq \mathscr{G}^{\omega}_{a,b}(x_0,h). \tag{8.10}$$

As a consequence, the pseudo-ball $\mathscr{G}^{\omega}_{a,b}(x_0,h)$ contains truncated circular cones (with vertex at x_0 and axis h) of apertures arbitrarily close to π.

Proof. These are all straightforward consequences of definitions. $\square$

For the remainder of this chapter we shall assume that the shape function ω as in (1.14) also satisfies the conditions listed in (1.16). That is (after a slight rephrasing of the first condition in (1.16)),

$$\begin{aligned} & R \in (0,+\infty) \ \text{ and } \ \omega : [0,R] \to [0,+\infty) \ \text{ is a continuous, (strictly)} \\ & \text{increasing function, with the property that } \omega(0) = 0 \text{ and there exists} \\ & \text{a function } \ \eta : (0,+\infty) \to (0,+\infty) \text{which satisfies } \lim_{\lambda \to 0^+} \eta(\lambda) = 0 \\ & \text{and } \ \omega(\lambda t) \leq \eta(\lambda)\,\omega(t) \text{ for all } \ \lambda > 0 \text{ and } t \in \big[0, \min\{R, R/\lambda\}\big]. \end{aligned} \tag{8.11}$$

For future reference, let us note here that conditions (8.11) entail that

$$\begin{aligned} & \omega : [0,R] \to [0,\omega(R)] \ \text{ is invertible and its inverse} \\ & \omega^{-1} : [0,\omega(R)] \to [0,R] \text{ is a continuous function which} \\ & \text{is (strictly) increasing and satisfies } \ \omega^{-1}(0) = 0. \end{aligned} \tag{8.12}$$

In order to facilitate the presentation of the proof of the main result in this paper we shall now present a series of technical, preliminary lemmas pertaining to the geometry of pseudo-balls. The key ingredient is the fact that a pseudo-ball has positive, finite curvature near the apex. A concrete manifestation of this property is the fact that two pseudo-balls with apex at the origin and whose direction vectors do not point in opposite ways necessarily have a substantial overlap.

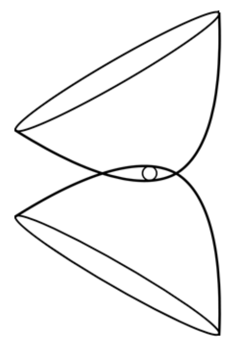


Figure 5. *Any two pseudo-balls with a common apex and whose direction vectors are not opposite contain a ball in their overlap (with quantitative control of its size).*

A precise, quantitative aspect of this phenomenon is discussed in Lemma 8.2 below.

Lemma 8.2. *Assume* (8.11) *and let $a, b \in (0, +\infty)$ be given. Then there exists $\varepsilon > 0$, which depends only on η, ω, R, a and b, such that for any $h_0, h_1 \in S^{n-1}$ the following implication holds:*

$$\begin{gathered} x \in \mathbb{R}^n \quad \textit{and} \quad \Big| x - \varepsilon\,\omega^{-1}\Big(\omega(R)\tfrac{|h_0+h_1|}{2}\Big)\,\tfrac{h_0+h_1}{|h_0+h_1|}\Big| < \tfrac{\varepsilon}{2}\,\omega^{-1}\Big(\omega(R)\tfrac{|h_0+h_1|}{2}\Big)\Big|\tfrac{h_0+h_1}{2}\Big| \Longrightarrow \\ |x| < R \quad \textit{and} \quad a|x|\omega(|x|) < \min\big\{x\cdot h_0, x\cdot h_1\big\} \quad \textit{and} \quad \max\big\{x\cdot h_0, x\cdot h_1\big\} < b, \end{gathered} \tag{8.13}$$

with the convention that $\frac{h_0+h_1}{|h_0+h_1|} := 0$ if $h_0 + h_1 = 0$. In other words, for each vectors $h_0, h_1 \in S^{n-1}$, the first line in (8.13) *implies that $x \in \mathscr{G}^{\omega}_{a,b}(0, h_0) \cap \mathscr{G}^{\omega}_{a,b}(0, h_1)$.*

Proof. Fix a real number ε such that

$$0 < \varepsilon < \min\Big\{\tfrac{2b}{3R}\,,\,\tfrac{2}{3}\Big\} \quad \text{and} \quad \eta\big(\tfrac{3\varepsilon}{2}\big) < \big[3a\,\omega(R)\big]^{-1}. \tag{8.14}$$

That this is possible is ensured by the last line in (8.11). Next, pick two vectors $h_0, h_1 \in S^{n-1}$ and introduce $v := \frac{h_0+h_1}{2}$. Then, if x is as in the first line in (8.13), we may estimate (keeping in mind that $|v| \leq 1$):

$$\begin{aligned} |x| &\leq \left|x - \varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\tfrac{v}{|v|}\right| + \left|\varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\tfrac{v}{|v|}\right| \\ &< \tfrac{\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big)|v| + \varepsilon\,\omega^{-1}\big(\omega(R)|v|\big) \leq \tfrac{3\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big). \end{aligned} \tag{8.15}$$

Granted the first condition in (8.14), this further implies (recall that ω^{-1} is increasing and $|v| \leq 1$) that

$$|x| \leq \tfrac{3\varepsilon}{2} R < \min\{R, b\} \tag{8.16}$$

so the first estimate in the second line of (8.13) is taken care of. In order to prove the remaining estimates in the second line of (8.13), it is enough to show that

$$\begin{aligned} &a|x|\omega(|x|) < x \cdot h_0 < b \ \text{ if } x \in \mathbb{R}^n \text{ is such that} \\ &\left|x - \varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\tfrac{v}{|v|}\right| < \tfrac{\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big)|v|, \end{aligned} \tag{8.17}$$

since the roles of h_0 and h_1 in (8.13) are interchangeable. To this end, assume that x is as in the last part of (8.17) and write

$$x \cdot h_0 = \Big(x - \varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\tfrac{v}{|v|}\Big) \cdot h_0 + \varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\tfrac{v}{|v|} \cdot h_0 \tag{8.18}$$

and observe that $v \cdot h_0 = \frac{1}{2}(1 + h_0 \cdot h_1) = |v|^2$. Thus, on the one hand, we have

$$\varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\tfrac{v}{|v|} \cdot h_0 = \varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)|v|. \tag{8.19}$$

On the other hand, based on the Cauchy-Schwarz inequality and the assumption on x we obtain

$$\left|\Big(x - \varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\tfrac{v}{|v|}\Big) \cdot h_0\right| \leq \left|x - \varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\tfrac{v}{|v|}\right| < \tfrac{\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big)|v|. \tag{8.20}$$

From this it follows that

$$x \cdot h_0 > \varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)|v| - \tfrac{\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big)|v| = \tfrac{\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big)|v|. \tag{8.21}$$

At this stage, we make the claim that $\frac{\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big)|v| > a|x|\omega(|x|)$ which, when used in concert with the estimate just derived, yields $x \cdot h_0 > a|x|\omega(|x|)$. To justify this claim, based on (8.15) and (8.11) we may then write

$$\begin{aligned} a|x|\omega(|x|) &\leq a\tfrac{3\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big)\,\omega\Big(\tfrac{3\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big)\Big) \\ &\leq a\tfrac{3\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big)\,\eta\big(\tfrac{3\varepsilon}{2}\big)\,\omega(R)|v| \\ &< \tfrac{\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big)|v|, \end{aligned} \tag{8.22}$$

where the third inequality is a consequence of (8.14). This finishes the proof of the claim. There remains to observe that, thanks to (8.16), $x \cdot h_0 \leq |x| < b$, completing the proof of Lemma 8.2. □

The main application of Lemma 8.2 is the following result asserting, in a quantitative manner, that two pseudo-balls necessarily overlap if their apexes are sufficiently close to one another relative to the degree of proximity of their axes.

Lemma 8.3. *Assume* (8.11) *and suppose that* $a, b \in (0, +\infty)$ *are given. Also, suppose that the parameter* $\varepsilon = \varepsilon(\omega, \eta, R, a, b) > 0$ *is as in Lemma 8.2. Then for every* $x_0, x_1 \in \mathbb{R}^n$ *and every* $h_0, h_1 \in S^{n-1}$ *one has*

$$|x_0 - x_1| < \tfrac{\varepsilon}{2}\,\omega^{-1}\Big(\omega(R)\tfrac{|h_0+h_1|}{2}\Big)\Big|\tfrac{h_0+h_1}{2}\Big| \Longrightarrow \mathscr{G}^{\omega}_{a,b}(x_0, h_0) \cap \mathscr{G}^{\omega}_{a,b}(x_1, h_1) \neq \varnothing. \tag{8.23}$$

Proof. To set the stage, let $\varepsilon > 0$ be as in Lemma 8.2 and assume that $x_0, x_1 \in \mathbb{R}^n$ and $h_0, h_1 \in S^{n-1}$ are such that the estimate in the left-hand side of (8.23) holds. In particular, $|h_0 + h_1| > 0$ and

$$B\Big(\varepsilon\,\omega^{-1}\Big(\omega(R)\tfrac{|h_0+h_1|}{2}\Big)\,\tfrac{h_0+h_1}{|h_0+h_1|}, \tfrac{\varepsilon}{2}\,\omega^{-1}\Big(\omega(R)\tfrac{|h_0+h_1|}{2}\Big)\Big|\tfrac{h_0+h_1}{2}\Big|\Big) \subseteq \mathscr{G}^{\omega}_{a,b}(0, h_0) \cap \mathscr{G}^{\omega}_{a,b}(0, h_1). \tag{8.24}$$

Indeed, this is simply a rephrasing of the conclusion in Lemma 8.2. Henceforth, we denote the ball in the left-hand side of (8.24) by $\mathcal{B}_{h_0,h_1}$ and denote its center and its radius by c_{h_0,h_1} and r_{h_0,h_1}, respectively. To proceed, consider $y := c_{h_0,h_1} + x_0 - x_1 \in \mathbb{R}^n$ and note that we may estimate

$|y - c_{h_0,h_1}| = |x_0 - x_1| < r_{h_0,h_1}$. This implies that $y \in \mathcal{B}_{h_0,h_1} \subseteq \mathscr{G}^{\omega}_{a,b}(0,h_1)$, thus, ultimately, $c_{h_0,h_1} = (x_1 - x_0) + y \in x_1 - x_0 + \mathscr{G}^{\omega}_{a,b}(0,h_1)$. Since we also have $c_{h_0,h_1} \in \mathcal{B}_{h_0,h_1} \subseteq \mathscr{G}^{\omega}_{a,b}(0,h_0)$, this analysis shows that $c_{h_0,h_1} \in \mathscr{G}^{\omega}_{a,b}(0,h_0) \cap \Big(x_1 - x_0 + \mathscr{G}^{\omega}_{a,b}(0,h_1)\Big)$. Upon recalling from item (iv) in Lemma 8.1 that $x_1 - x_0 + \mathscr{G}^{\omega}_{a,b}(0,h_1) = \mathscr{G}^{\omega}_{a,b}(x_1 - x_0, h_1)$, we deduce that $\mathscr{G}^{\omega}_{a,b}(0,h_0) \cap \mathscr{G}^{\omega}_{a,b}(x_1 - x_0, h_1)$ is nonempty. Finally, translating by x_0 yields $\mathscr{G}^{\omega}_{a,b}(x_0,h_0) \cap \mathscr{G}^{\omega}_{a,b}(x_1,h_1) \neq \varnothing$. This completes the proof of Lemma 8.3. $\square$

We conclude this chapter by presenting a consequence of Lemma 8.3 to the effect that two pseudo-balls sharing a common apex are disjoint if and only if their axes point in opposite directions.

Corollary 8.4. *Assume* (8.11) *and suppose that* $a, b \in (0, +\infty)$ *are given. Then for each point* $x \in \mathbb{R}^n$ *and any pair of vectors* $h_0, h_1 \in S^{n-1}$ *one has*

$$\mathscr{G}^{\omega}_{a,b}(x,h_0) \cap \mathscr{G}^{\omega}_{a,b}(x,h_1) = \varnothing \Longleftrightarrow h_0 + h_1 = 0. \tag{8.25}$$

Proof. If $x \in \mathbb{R}^n$ and $h_0, h_1 \in S^{n-1}$ are such that $|h_0 + h_1| > 0$ and yet $\mathscr{G}^{\omega}_{a,b}(x,h_0) \cap \mathscr{G}^{\omega}_{a,b}(x,h_1) = \varnothing$, then Lemma 8.3 (used with $x_0 := x =: x_1$) yields a contradiction. This proves the left-to-right implication in (8.25). For the converse implication, observe that if $y \in \mathscr{G}^{\omega}_{a,b}(x,h) \cap \mathscr{G}^{\omega}_{a,b}(x,-h)$ for some $x \in \mathbb{R}^n$ and $h \in S^{n-1}$ then $a|y-x|\,\omega(|y-x|) < h \cdot (y-x)$ and $a|y-x|\,\omega(|y-x|) < (-h) \cdot (y-x)$. Hence $h \cdot (y - x) > 0$ and $(-h) \cdot (y - x) > 0$, a contradiction which concludes the proof of the corollary. $\square$

Chapter 9

Sets of Locally Finite Perimeter

Given $E \subseteq \mathbb{R}^n$, denote by $\mathbf{1}_E$ the characteristic function of E. A Lebesgue measurable set $E \subseteq \mathbb{R}^n$ is said to be of locally finite perimeter provided

$$\mu := \nabla \mathbf{1}_E \tag{9.1}$$

is a locally finite $\mathbb{R}^n$-valued measure. For a set of locally finite perimeter which has a compact boundary we agree to drop the adverb 'locally'. Given a set $E \subseteq \mathbb{R}^n$ of locally finite perimeter we denote by σ the total variation measure of μ; σ is then a locally finite positive measure supported on ∂E, the topological boundary of E. Also, clearly, each component of μ is absolutely continuous with respect to σ. It follows from the Radon-Nikodym theorem that

$$\mu = \nabla \mathbf{1}_E = -\nu\sigma, \tag{9.2}$$

where

$$\begin{aligned} &\nu \in L^\infty(\partial E, d\sigma) \text{ is an } \mathbb{R}^n\text{-valued function} \\ &\text{satisfying } |\nu(x)| = 1, \text{ for } \sigma\text{-a.e. } x \in \partial E. \end{aligned} \tag{9.3}$$

It is customary to identify σ with its restriction to ∂E with no special mention. We shall refer to ν and σ, respectively, as the (geometric measure theoretic) outward unit normal and the surface measure on ∂E. Note that ν defined by (9.2) can only be specified up to a set of σ-measure zero.

To eliminate this ambiguity, we redefine $\nu(x)$, for every x, as being

$$\lim_{r\to 0} \fint_{B(x,r)} \nu \, d\sigma \tag{9.4}$$

whenever the above limit exists, and zero otherwise. In doing so, we shall make the convention that $\fint_{B(x,r)} \nu \, d\sigma := (\sigma(B(x,r)))^{-1} \int_{B(x,r)} \nu \, d\sigma$ if $\sigma(B(x,r)) > 0$, and zero otherwise. The Besicovitch Differentiation Theorem (cf., e.g., [21]) ensures that ν in (9.2) agrees with (9.4) for σ-a.e. x.

The reduced boundary of E is then defined as

$$\partial^* E := \big\{ x \in \partial E : \, |\nu(x)| = 1 \big\}. \tag{9.5}$$

This is essentially the point of view adopted in [93] (cf. Definition 5.5.1 on p. 233). Let us remark that this definition is slightly different from that given on p. 194 of [21]. The reduced boundary introduced there depends on the choice of the unit normal in the class of functions agreeing with it σ-a.e. and, consequently, can be pointwise specified only up to a certain set of zero surface measure. Nonetheless, any such representative is a subset of $\partial^* E$ defined above and differs from it by a set of σ-measure zero.

Moving on, it follows from (9.5) and the Besicovitch Differentiation Theorem that σ is supported on $\partial^* E$, in the sense that $\sigma(\mathbb{R}^n \setminus \partial^* E) = 0$. From the work of Federer and of De Giorgi it is also known that, if $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$,

$$\sigma = \mathcal{H}^{n-1} \lfloor \partial^* E. \tag{9.6}$$

Recall that, generally speaking, given a Radon measure μ in $\mathbb{R}^n$ and a set $A \subseteq \mathbb{R}^n$, the restriction of μ to A is defined as $\mu \lfloor A := \mathbf{1}_A \, \mu$. In particular, $\mu \lfloor A \ll \mu$ and $d(\mu \lfloor A)/d\mu = \mathbf{1}_A$. Thus,

$$\sigma \ll \mathcal{H}^{n-1} \quad \text{and} \quad \frac{d\sigma}{d\mathcal{H}^{n-1}} = \mathbf{1}_{\partial^* E}. \tag{9.7}$$

Furthermore (cf. Lemma 5.9.5 on p. 252 in [93], and p. 208 in [21]) one has

$$\partial^* E \subseteq \partial_* E \subseteq \partial E, \quad \text{and} \quad \mathcal{H}^{n-1}(\partial_* E \setminus \partial^* E) = 0, \tag{9.8}$$

where $\partial_* E$, the *measure-theoretic boundary* of E, is defined by

$$\partial_* E := \Big\{ x \in \partial E : \limsup_{r \to 0^+} r^{-n} \mathcal{H}^n\big(B(x,r) \cap E^{\pm}\big) > 0 \Big\}. \tag{9.9}$$

Above, $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure (i.e., up to normalization, the Lebesgue measure) in $\mathbb{R}^n$, and we have set $E^+ := E$, $E^- := \mathbb{R}^n \setminus E$.

Let us also record here a useful criterion for deciding whether a Lebesgue measurable subset E of $\mathbb{R}^n$ is of locally finite perimeter in $\mathbb{R}^n$ (cf. [21], p. 222):

$$E \text{ has locally finite perimeter} \iff \mathcal{H}^{n-1}(\partial_* E \cap K) < +\infty, \quad \forall K \subseteq \mathbb{R}^n \text{ compact.} \tag{9.10}$$

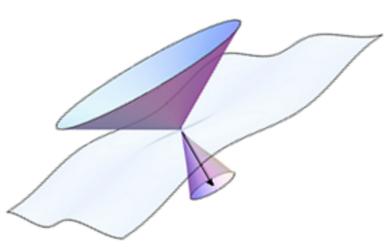
We conclude this chapter by proving the following result of geometric measure theoretic flavor (which is a slight extension of Proposition 2.9 in [40]), establishing a link between the cone property and the direction of the geometric measure theoretic unit normal.

Lemma 9.1. *Let E be a subset of $\mathbb{R}^n$ of locally finite perimeter. Fix a point x_0 belonging to $\partial^* E$ (the reduced boundary of E) with the property that there exist $b > 0$, $\theta \in (0, \pi)$ and $h \in S^{n-1}$ such that*

$$\Gamma_{\theta,b}(x_0, h) \subseteq E. \tag{9.11}$$

Then, if $\nu(x_0)$ denotes the geometric measure theoretic outward unit normal to E at x_0, there holds

$$\nu(x_0) \in \Gamma_{\pi-\theta,1}(0, -h). \tag{9.12}$$



Proof. The idea is to use a blow-up argument. Specifically, consider the half-space

$$H(x_0) := \{x \in \mathbb{R}^n : \nu(x_0) \cdot (x - x_0) < 0\} \subseteq \mathbb{R}^n \tag{9.13}$$

and, for each $r > 0$ and $A \subseteq \mathbb{R}^n$, set

$$A_r := \{x \in \mathbb{R}^n : r(x - x_0) + x_0 \in A\}. \tag{9.14}$$

Also, abbreviate $\Gamma := \Gamma_{\theta,b}(x_0, h)$ and denote by $\widetilde{\Gamma}$ the circular, open, infinite cone which coincides with $\Gamma_{\theta,b}(x_0, h)$ near its vertex. The theorem concerning the blow-up of the reduced boundary of a set of locally finite perimeter (cf., e.g., p. 199 in [21]) gives that

$$\mathbf{1}_{E_r} \longrightarrow \mathbf{1}_{H(x_0)} \quad \text{in} \quad L^1_{loc}(\mathbb{R}^n), \quad \text{as} \quad r \to 0^+. \tag{9.15}$$

On the other hand, it is clear that $\Gamma_r \subseteq E_r$ and $\mathbf{1}_{\Gamma_r} \longrightarrow \mathbf{1}_{\widetilde{\Gamma}}$ in $L^1_{loc}(\mathbb{R}^n)$ as $r \to 0^+$. This and (9.14) then allow us to write

$$\begin{aligned} \mathbf{1}_{\widetilde{\Gamma}} &= \lim_{r\to 0^+} \mathbf{1}_{\Gamma_r} = \lim_{r\to 0^+} \Big(\mathbf{1}_{\Gamma_r} \cdot \mathbf{1}_{E_r}\Big) \\ &= \Big(\lim_{r\to 0^+} \mathbf{1}_{\Gamma_r}\Big) \cdot \Big(\lim_{r\to 0^+} \mathbf{1}_{E_r}\Big) = \mathbf{1}_{\widetilde{\Gamma}} \cdot \mathbf{1}_{H(x_0)} \\ &= \mathbf{1}_{\widetilde{\Gamma}\cap H(x_0)}, \end{aligned} \tag{9.16}$$

in a pointwise $\mathcal{H}^n$-a.e. sense in $\mathbb{R}^n$. In turn, this implies

$$\widetilde{\Gamma} \subseteq \overline{H(x_0)}. \tag{9.17}$$

Now, (9.12) readily follows from this, (9.13), and simple geometrical considerations. $\square$

Chapter 10

Measuring the Smoothness of Euclidean Domains in Analytical Terms

We begin by giving the formal definition of the category of Lipschitz domains and domains of class $\mathscr{C}^{1,\alpha}$, $\alpha \in (0,1]$. The reader is reminded that the superscript c is the operation of taking the complement of a set, relative to $\mathbb{R}^n$.

Definition 10.1. *Let Ω be a nonempty, proper, open subset of $\mathbb{R}^n$. Also, fix $x_0 \in \partial\Omega$. Call Ω a* `Lipschitz domain near` *x_0 if there exist $r, c > 0$ with the following significance. There exist an $(n-1)$-dimensional plane $H \subseteq \mathbb{R}^n$ passing through the point x_0, a choice N of the unit normal to H, and an open cylinder $\mathcal{C}_{r,c} := \{x' + tN : x' \in H, |x' - x_0| < r, |t| < c\}$ (called coordinate cylinder near x_0) such that*

$$\mathcal{C}_{r,c} \cap \Omega = \mathcal{C}_{r,c} \cap \{x' + tN : x' \in H, \ t > \varphi(x')\}, \tag{10.1}$$

for some Lipschitz function $\varphi : H \to \mathbb{R}$, called the defining function for $\partial\Omega$ near x_0, satisfying

$$\varphi(x_0) = 0 \quad \text{and} \quad |\varphi(x')| < c \quad \text{if} \quad |x' - x_0| \leq r. \tag{10.2}$$

Collectively, the pair $(\mathcal{C}_{r,c}, \varphi)$ will be referred to as a local chart near x_0, whose geometrical characteristics consist of r, c and the Lipschitz constant of φ.

Moreover, call Ω a `locally Lipschitz domain` *if it is a Lipschitz domain near every point $x \in \partial\Omega$. Finally, Ω is simply called a* `Lipschitz domain` *if it is locally Lipschitz and such that the geometrical characteristics of the local charts associated with each boundary point are independent of the point in question.*

The categories of $\mathscr{C}^{1,\alpha}$ domains with $\alpha \in (0,1]$, as well as their local versions, are defined analogously, requiring that the defining functions φ have first order directional derivatives (along vectors parallel to the hyperplane H) which are of class $\mathscr{C}^{\alpha}$ (the Hölder space of order α).

A few useful observations related to the property of an open set $\Omega \subseteq \mathbb{R}^n$ of being a Lipschitz domain near a point $x_0 \in \partial\Omega$ are collected below.

Proposition 10.1. *Assume that Ω is a nonempty, proper, open subset of $\mathbb{R}^n$, and fix $x_0 \in \partial\Omega$.*

(i) If Ω is a Lipschitz domain near x_0 and if $(\mathcal{C}_{r,c}, \varphi)$ is a local chart near x_0 (in the sense of Definition 10.1) then, in addition to (10.1)*, one also has*

$$\mathcal{C}_{r,c} \cap \partial\Omega = \mathcal{C}_{r,c} \cap \{x' + tN : \, x' \in H, \; t = \varphi(x')\}, \tag{10.3}$$

$$\mathcal{C}_{r,c} \cap (\overline{\Omega})^c = \mathcal{C}_{r,c} \cap \{x' + tN : \, x' \in H, \; t < \varphi(x')\}. \tag{10.4}$$

Furthermore,

$$\mathcal{C}_{r,c} \cap \overline{\Omega} = \mathcal{C}_{r,c} \cap \{x' + tN : \, x' \in H, \; t \geq \varphi(x')\}, \tag{10.5}$$

$$\mathcal{C}_{r,c} \cap (\overline{\Omega})^{\circ} = \mathcal{C}_{r,c} \cap \{x' + tN : \, x' \in H, \; t > \varphi(x')\}, \tag{10.6}$$

and, consequently,

$$E \cap \partial\Omega = E \cap \partial(\overline{\Omega}), \qquad \forall\, E \subseteq \mathcal{C}_{r,c}. \tag{10.7}$$

(ii) Assume that there exist an $(n-1)$-dimensional plane $H \subseteq \mathbb{R}^n$ passing through x_0, a choice N of the unit normal to H, an open cylinder $\mathcal{C}_{r,c} := \{x' + tN : x' \in H, |x' - x_0| < r, |t| < c\}$ and a Lipschitz function $\varphi : H \to \mathbb{R}$ satisfying (10.2) such that (10.3) holds. Then, if $x_0 \notin (\overline{\Omega})^\circ$, it follows that Ω is a Lipschitz domain near x_0.

Proof. The fact that (10.1) implies (10.3) is a consequence of the general fact

$$\mathcal{O}, \Omega_1, \Omega_2 \subseteq \mathbb{R}^n \text{ open sets such that } \mathcal{O} \cap \Omega_1 = \mathcal{O} \cap \Omega_2 \Longrightarrow \mathcal{O} \cap \partial\Omega_1 = \mathcal{O} \cap \partial\Omega_2, \tag{10.8}$$

used with $\mathcal{O} := \mathcal{C}_{r,c}$, $\Omega_1 := \Omega$ and Ω_2 the upper-graph of φ. In order to justify (10.8), we make the elementary observation that

$$E \subseteq \mathbb{R}^n \text{ arbitrary set and } \mathcal{O} \subseteq \mathbb{R}^n \text{ open set } \Longrightarrow \overline{E} \cap \mathcal{O} \subseteq \overline{E \cap \Omega}. \tag{10.9}$$

Then, in the context of (10.8), based on assumptions and (10.9) we may write

$$\begin{aligned} \mathcal{O} \cap \partial\Omega_1 \subseteq (\mathcal{O} \cap \overline{\Omega_1}) \setminus (\mathcal{O} \cap \Omega_1) \quad &\subseteq \quad \overline{\mathcal{O} \cap \Omega_1} \setminus (\mathcal{O} \cap \Omega_1) \\ &= \quad \overline{\mathcal{O} \cap \Omega_2} \setminus (\mathcal{O} \cap \Omega_2) \subseteq \overline{\Omega_2} \setminus \Omega_2 = \partial\Omega_2. \end{aligned} \tag{10.10}$$

This further entails $\mathcal{O} \cap \partial\Omega_1 \subseteq \mathcal{O} \cap \partial\Omega_2$ from which (10.8) follows by interchanging the roles of Ω_1 and Ω_2. As mentioned earlier, this establishes (10.3). Thus, in order to prove (10.4), it suffices to show that

$$\text{(10.1) and (10.3)} \Longrightarrow \text{(10.4)}, \tag{10.11}$$

In turn, (10.11) follows by writing

$$\begin{aligned}
\mathcal{C}_{r,c} \cap (\overline{\Omega})^c &= \mathcal{C}_{r,c} \setminus (\mathcal{C}_{r,c} \cap \overline{\Omega}) = \mathcal{C}_{r,c} \setminus \Big((\mathcal{C}_{r,c} \cap \Omega) \cup (\mathcal{C}_{r,c} \cap \partial\Omega) \Big) \\
&= \mathcal{C}_{r,c} \setminus \Big((\mathcal{C}_{r,c} \cap \{x' + tN : x' \in H,\ t > \varphi(x')\}) \\
&\qquad\qquad\qquad \cup (\mathcal{C}_{r,c} \cap \{x' + tN : x' \in H,\ t = \varphi(x')\}) \Big) \\
&= \mathcal{C}_{r,c} \setminus \Big(\mathcal{C}_{r,c} \cap \{x' + tN : x' \in H,\ t \geq \varphi(x')\} \Big) \\
&= \mathcal{C}_{r,c} \cap \{x' + tN : x' \in H,\ t < \varphi(x')\}, \qquad (10.12)
\end{aligned}$$

as desired. Next, (10.5) is a consequence of (10.1) and (10.3), while (10.6) follows from (10.5) by passing to interiors. In concert, (10.5)-(10.6) and (10.3) give that

$$\mathcal{C}_{r,c} \cap \partial(\overline{\Omega}) = \mathcal{C}_{r,c} \cap \partial\Omega, \qquad (10.13)$$

which further implies (10.7) by taking the intersection of both sides with a given set $E \subseteq \mathcal{C}_{r,c}$. This completes the proof of part (i). As far as (ii) is concerned, it suffices to show that, up to reversing the sense on the vertical axis in $\mathbb{R}^{n-1} \times \mathbb{R}$,

$$x_0 \notin (\overline{\Omega})^\circ \implies (10.1),\ (10.4). \qquad (10.14)$$

In turn, (10.14) follows from Lemma 10.2, stated and proved below. □

Here is the topological result which has been invoked earlier, in the proof of the implication (10.14).

Lemma 10.2. *Assume that $\Omega \subseteq \mathbb{R}^n$ is a nonempty, proper, open set, and fix $x_0 \in \partial\Omega$. Also, assume that $B' \subseteq \mathbb{R}^{n-1}$ is an $(n-1)$-dimensional open ball, $I \subseteq \mathbb{R}$ is an open interval, and that $\varphi : B' \to I$ is a continuous function. Denote the graph of φ by $\Sigma := \{(x', \varphi(x')) : x' \in B'\}$. Assume that the* *open cylinder* $\mathcal{C} := B' \times I \subseteq \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$ *contains* x_0 *and satisfies* $\Sigma = \mathcal{C} \cap \partial\Omega$. *Finally, set*

$$D^+ := \{(x', x_n) \in \mathcal{C} : \varphi(x') < x_n\}, \quad D^- := \{(x', x_n) \in \mathcal{C} : \varphi(x') > x_n\}. \tag{10.15}$$

Then one of the following three alternatives holds:

$$\Omega \cap \mathcal{C} = D^+ \quad \textit{and} \quad (\overline{\Omega})^c \cap \mathcal{C} = D^-, \tag{10.16}$$

$$\Omega \cap \mathcal{C} = D^- \quad \textit{and} \quad (\overline{\Omega})^c \cap \mathcal{C} = D^+, \tag{10.17}$$

$$x_0 \in (\overline{\Omega})^\circ. \tag{10.18}$$

Proof. We begin by noting that $D^\pm$ are connected sets. To see this, consider D^+, as the argument for D^- is similar. It suffices to show that the set in question is pathwise connected, and a continuous curve γ contained in D^+ joining any two given points $x, y \in D^+$ may be taken to consist of three line segments, L_1, L_2, L_3, defined as follows. Take L_1 and L_2 to be the vertical line segments contained in D^+ which emerge from x and y, respectively, and then choose L_3 to be a horizontal line segment making the transition between L_1 and L_2 near the very top of $\mathcal{C}$. Moving on, we claim that one of the following situations necessarily happens:

$$\begin{aligned} &(i)\ \ D^+ \subseteq \Omega \quad \text{and} \quad D^- \subseteq (\overline{\Omega})^c, \qquad \text{or} \\ &(ii)\ \ D^- \subseteq \Omega \quad \text{and} \quad D^+ \subseteq (\overline{\Omega})^c, \qquad \text{or} \\ &(iii)\ \ D^+ \subseteq \Omega \quad \text{and} \quad D^- \subseteq \Omega, \qquad \text{or} \\ &(iv)\ \ D^- \subseteq (\overline{\Omega})^c \quad \text{and} \quad D^+ \subseteq (\overline{\Omega})^c. \end{aligned} \tag{10.19}$$

To prove this, note that $D^\pm$ are disjoint from Σ and, hence, from $\partial\Omega \cap \mathcal{C}$. In turn, this further entails that $D^\pm$ are disjoint from $\partial\Omega$. Based on this and the fact that $\mathbb{R}^n = \Omega \cup (\overline{\Omega})^c \cup \partial\Omega$, we conclude that

$$D^\pm \subseteq \Omega \cup (\overline{\Omega})^c. \tag{10.20}$$

Now, recall that $D^\pm$ are connected sets, and observe that Ω are $(\overline{\Omega})^c$ open, disjoint sets. In concert with the definition of connectivity, (10.20) then implies that each of the two sets D^+, D^- is contained in either Ω, or $(\overline{\Omega})^c$. Unraveling the various possibilities now proves that one of the four scenarios in (10.19) must hold. This concludes the proof of the claim made about (10.19). The next step is to show that if conditions (i) in (10.19) happen, then conditions (10.16) happen as well. To see this, assume (i) holds, i.e., $D^+ \subseteq \Omega$, $D^- \subseteq (\overline{\Omega})^c$ and recall that $\Sigma = \partial\Omega \cap \mathcal{C}$. Then $D^+ = \Omega \cap \mathcal{C}$. Indeed, the left-to-right inclusion is clear from what we assume. For the opposite inclusion, we reason by contradiction and assume that there exists $x \in \Omega \cap \mathcal{C}$ such that $x \notin D^+$, thus $x \in \Omega$, $x \in \mathcal{C}$, $x \notin D^+$. Since

$$\mathcal{C} = D^+ \cup D^- \cup \Sigma, \quad \text{disjoint unions,} \tag{10.21}$$

we obtain

$$\mathcal{C} = D^+ \cup D^- \cup (\mathcal{C} \cap \partial\Omega), \quad \text{disjoint unions.} \tag{10.22}$$

From the assumptions on x we have that $x \notin D^+$, $x \notin \mathcal{C} \cap \partial\Omega$ (since $x \in \Omega$ and $\Omega \cap \partial\Omega = \varnothing$) and consequently, using also (10.22), $x \in D^- \subseteq (\overline{\Omega})^c$. This yields that $x \notin \Omega$, contradicting the assumption that $x \in \Omega$. This completes the proof of the fact that $D^+ = \Omega \cap \mathcal{C}$. In a similar fashion, we also obtain that $D^- = (\overline{\Omega})^c \cap \mathcal{C}$.

We next propose to show that if condition (ii) in (10.19) happens, then condition (10.17) happens as well. In particular, if (ii) happens, then

$$\Omega \cap \mathcal{C} = D^-, \quad (\overline{\Omega})^c \cap \mathcal{C} = D^+, \quad \partial\Omega \cap \mathcal{C} = \Sigma. \tag{10.23}$$

To see this, assume (ii) happens. The first observation is that $D^+ = (\overline{\Omega})^c \cap \mathcal{C}$. The left-to-right inclusion is clear. Assume next that there exists $x \in \mathcal{C}$ such that $x \notin \overline{\Omega}$ (thus $x \notin \partial\Omega$) and $x \notin D^+$. Together with (10.22), these imply $x \in D^-$ so $x \in D^- \subseteq \Omega \subseteq \overline{\Omega}$. which is in contradiction with our assumptions. Moving on, the second observation is that $D^- = \Omega \cap \mathcal{C}$. The left-to-right inclusion is obvious. In the opposite direction, assume that there exists $x \in \mathcal{C}$ such that $x \in \Omega$ (hence $x \notin \partial\Omega$) yet $x \notin D^-$. Invoking (10.22) it follows that $x \in D^+ \subseteq (\overline{\Omega})^c$, hence $x \notin \overline{\Omega}$ contradicting the assumption on x.

Next, we shall show that if condition (iii) in (10.19) happens, then $x_0 \in (\overline{\Omega})^\circ \cap \partial\Omega$, i.e. (10.18) happens. To this end, let $x_* \in \partial\Omega \cap \mathcal{C}$. Then there exists $r > 0$ such that $B(x_*, r) \subseteq \mathcal{C}$. We claim that

$$B(x_*, r) \subseteq \overline{\Omega} \tag{10.24}$$

Indeed, by (10.21), we have

$$B(x_*, r) = \Big(B(x_*, r) \cap D^+\Big) \cup \Big(B(x_*, r) \cap D^-\Big) \cup \Big(B(x_*, r) \cap \Sigma\Big). \tag{10.25}$$

Making use of the inclusion $B(x_*, r) \subseteq \mathcal{C}$, we then obtain

$$\begin{aligned} &B(x_*, r) \cap D^+ \subseteq D^+ \subseteq \Omega, \\ &B(x_*, r) \cap D^- \subseteq D^- \subseteq \Omega, \\ &B(x_*, r) \cap \Sigma \subseteq \Sigma = \mathcal{C} \cap \partial\Omega \subseteq \partial\Omega. \end{aligned} \tag{10.26}$$

Combining all these with (10.25), it follows that $B(x_*, r) \subseteq \Omega \cap \partial\Omega = \overline{\Omega}$, proving (10.24). In turn, (10.24) implies that $x_* \in (\overline{\Omega})^\circ \cap \partial\Omega$ so that, ultimately,

$$\mathcal{C} \cap \partial\Omega \subseteq (\overline{\Omega})^\circ \cap \partial\Omega. \tag{10.27}$$

Since $x_0 \in \mathcal{C} \cap \partial\Omega$, this forces $x_0 \in (\overline{\Omega})^\circ \cap \partial\Omega$, as claimed.

At this stage in the proof, there remains to show that condition (vi) in (10.19) never happens. Reasoning by assume (iv) actually does happen, i.e.,

$$D^- \subseteq (\overline{\Omega})^c, \qquad D^+ \subseteq (\overline{\Omega})^c \quad \text{and} \quad \Sigma = \partial\Omega \cap \mathcal{C}. \tag{10.28}$$

Taking the union of the first two inclusions above yields

$$D^{+} \cup D^{-} \subseteq (\overline{\Omega})^{c} \Longrightarrow \mathcal{C} \setminus \Sigma \subseteq (\overline{\Omega})^{c} \Longrightarrow \mathcal{C} \cap (\Sigma^{c}) \subseteq (\overline{\Omega})^{c} \Longrightarrow \overline{\Omega} \subseteq \Sigma \cup \mathcal{C}^{c}, \tag{10.29}$$

where the last implication follows by taking complements. Taking the intersection with $\mathcal{C}$, this yields $\mathcal{C} \cap \overline{\Omega} \subseteq \Sigma = \mathcal{C} \cap \partial\Omega$, thanks to the fact that $\Sigma = \mathcal{C} \cap \partial\Omega$. Thus, $\mathcal{C} \cap \Omega \subseteq \mathcal{C} \cap \overline{\Omega} \subseteq \mathcal{C} \cap \partial\Omega \subseteq \partial\Omega$, i.e.,

$$\mathcal{C} \cap \Omega \subseteq \partial\Omega. \tag{10.30}$$

Since, by assumption, $\mathcal{C}$ is an open neighborhood of the point $x_0 \in \partial\Omega$, the definition of the boundary implies that $\mathcal{C} \cap \Omega \neq \varnothing$. Therefore, there exists $x_* \in \mathcal{C} \cap \Omega$. From (10.30) it follows that $x_* \in \partial\Omega$, which forces us to conclude that the open set Ω contains some of its own boundary points. This is a contradiction which shows that *(iv)* in (10.19) never happens. The proof of the lemma is therefore complete. □

The proposition below formalizes the idea that a connected, proper, open subset of $\mathbb{R}^n$ whose boundary is a compact Lipschitz surface is a Lipschitz domain. Before stating this, we wish to note that the connectivity assumption is necessary since, otherwise, $\Omega := \{x \in \mathbb{R}^n : |x| < 2 \text{ and } |x| \neq 1\}$ would serve as a counterexample.

Theorem 10.3. *Let Ω be a nonempty, connected, proper, open subset of $\mathbb{R}^n$, with $\partial\Omega$ bounded. In addition, suppose that for each $x_0 \in \partial\Omega$ there exist an $(n-1)$-dimensional plane $H \subseteq \mathbb{R}^n$ passing through x_0, a choice N of the unit normal to H, an open cylinder $\mathcal{C}_{r,c}$ and a Lipschitz function $\varphi : H \to \mathbb{R}$ satisfying* (10.2) *such that* (10.3) *holds. Then Ω is a Lipschitz domain.*

In the proof of the above result, the following generalization of the Jordan-Brouwer separation theorem for arbitrary compact topological hypersurfaces in $\mathbb{R}^n$, established in [7, Theorem 1, p. 284], plays a key role.

Proposition 10.4. *Let Σ be a compact, connected, topological $(n-1)$-dimensional submanifold (without boundary) of $\mathbb{R}^n$. Then $\mathbb{R}^n \setminus \Sigma$ consists of two connected components, each having Σ as their boundary.*

Proof of Theorem 10.3. Let Σ be a connected component of $\partial\Omega$ (in the relative topology induced by $\mathbb{R}^n$ on $\partial\Omega$). We claim that

$$\Sigma \subseteq \partial(\overline{\Omega}). \tag{10.31}$$

To justify this claim, we first observe that, granted the current assumptions, it follows that $\partial\Omega$ is a compact, $(n-1)$-dimensional Lipschitz sub-manifold (without boundary) of $\mathbb{R}^n$. Hence, in particular, Σ is a compact, connected, $(n-1)$-dimensional topological manifold (without boundary in $\mathbb{R}^n$). Invoking Proposition 10.4 we may then conclude that there exist $\mathcal{O}_1, \mathcal{O}_2 \subseteq \mathbb{R}^n$ such that

$$\begin{aligned} &\mathbb{R}^n \setminus \Sigma = \mathcal{O}_1 \cup \mathcal{O}_2 \quad \text{and} \quad \mathcal{O}_1 \cap \mathcal{O}_2 = \varnothing, \\ &\mathcal{O}_j \ \text{open, connected, } \ \partial\mathcal{O}_j = \Sigma \ \text{ for } \ j = 1, 2. \end{aligned} \tag{10.32}$$

Let us also observe that these conditions further entail

$$\partial(\overline{\mathcal{O}_j}) = \Sigma \ \text{ for } \ j = 1, 2. \tag{10.33}$$

Indeed, $\overline{\mathcal{O}_1} = \mathcal{O}_1 \cup \partial\mathcal{O}_1 = \mathcal{O}_1 \cup \Sigma = (\mathcal{O}_2)^c$ which forces $\partial(\overline{\mathcal{O}_1}) = \partial[(\mathcal{O}_2)^c] = \partial\mathcal{O}_2 = \Sigma$, from which (10.33) follows. Moreover, since Ω is a connected set contained in $\mathbb{R}^n \setminus \partial\Omega \subseteq \mathbb{R}^n \setminus \Sigma = \mathcal{O}_1 \cup \mathcal{O}_2$, it follows that Ω is contained in one of the sets $\mathcal{O}_1, \mathcal{O}_2$. To fix ideas, assume that $\Omega \subseteq \mathcal{O}_1$. Then $\overline{\Omega} \subseteq \overline{\mathcal{O}_1}$ and, hence,

$$(\overline{\Omega})^\circ \subseteq (\overline{\mathcal{O}_1})^\circ = \overline{\mathcal{O}_1} \setminus \partial(\overline{\mathcal{O}_1}) = (\mathcal{O}_1 \cup \partial\mathcal{O}_1) \setminus \partial\mathcal{O}_1 = \mathcal{O}_1, \tag{10.34}$$

where the next-to-last equality is a consequence of (10.33) (and (10.32)). The relevant observation for us here is that, in concert with the second line in (10.32), the inclusion in (10.34) forces

$$\Sigma \cap (\overline{\Omega})^\circ = \varnothing. \tag{10.35}$$

To proceed, note that since $\Sigma \subseteq \partial\Omega \subseteq \overline{\Omega}$, we also have

$$\Sigma \cap (\overline{\Omega})^c = \varnothing. \tag{10.36}$$

Thus, since

$$\Sigma \subseteq \mathbb{R}^n = (\overline{\Omega})^\circ \cup \partial(\overline{\Omega}) \cup (\overline{\Omega})^c, \tag{10.37}$$

we may ultimately deduce from (10.35)-(10.37) that (10.31) holds. The end-game in the proof of the proposition is then as follows. Taking the union of all connected components of $\partial\Omega$, we see from (10.31) that $\partial\Omega \subseteq \partial(\overline{\Omega})$. Consequently, since the opposite inclusion is always true, we arrive at the conclusion that

$$\partial\Omega = \partial(\overline{\Omega}). \tag{10.38}$$

Therefore, $\partial\Omega \cap (\overline{\Omega})^\circ = \partial(\overline{\Omega}) \cap (\overline{\Omega})^\circ = \varnothing$ and, as such, given any $x_0 \in \partial\Omega$ it follows that necessarily $x_0 \notin (\overline{\Omega})^\circ$. With this in hand, the fact that Ω is a Lipschitz domain now follows from part (*ii*) of Proposition 10.1. $\square$

Definition 10.1 and (*i*) in Proposition 10.1 show that if $\Omega \subseteq \mathbb{R}^n$ is a Lipschitz domain near a boundary point x_0 then, in a neighborhood of x_0, $\partial\Omega$ agrees with the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, considered in a suitably chosen system of coordinates (which is isometric with the original one). Then the outward unit normal has an explicit formula in terms of $\nabla\varphi$, namely, in the new system of coordinates,

$$\nu(x', \varphi(x')) = \frac{(\nabla'\varphi(x'), -1)}{\sqrt{1 + |\nabla'\varphi(x')|^2}}, \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x' \text{ near } x_0', \tag{10.39}$$

where the gradient $\nabla\varphi(x')$ of φ exists by the classical Rademacher theorem for $\mathcal{H}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$. This readily implies that if $\Omega \subseteq \mathbb{R}^n$ is a $\mathscr{C}^{1,\alpha}$ domain for some $\alpha \in (0, 1]$ then the outward unit normal $\nu : \partial\Omega \to S^{n-1}$ is Hölder of order α.

We next discuss a cone property enjoyed by Lipschitz domains whose significance will become more apparent later.

Lemma 10.5. *Assume that $\Omega \subseteq \mathbb{R}^n$ is Lipschitz near $x_0 \in \partial\Omega$. More specifically, suppose that the $(n-1)$-dimensional plane $H \subseteq \mathbb{R}^n$ passing through the point x_0, the unit normal N to H, the Lipschitz function $\varphi : H \to \mathbb{R}$ and the cylinder $\mathcal{C}_{r,c}$ are such that* (10.1)-(10.2) *hold. Denote by M the Lipschitz constant of φ and fix $\theta \in (0, 2\arctan(\frac{1}{M})]$. Finally, select $\lambda \in (0,1)$. Then there exists $b > 0$ such that*

$$\Gamma_{\theta,b}(x, N) \subseteq \Omega \quad \textit{and} \quad \Gamma_{\theta,b}(x, -N) \subseteq \mathbb{R}^n \setminus \Omega \quad \textit{for each } \; x \in \mathcal{C}_{\lambda r,c} \cap \partial\Omega. \tag{10.40}$$

Proof. Let $\theta \in \big(0, 2\arctan(\frac{1}{M})\big]$, where $M > 0$ is the Lipschitz constant of φ, and pick $b > 0$ such that

$$b < \min\Big\{c, \tfrac{(1-\lambda)r}{\tan(\theta/2)}\Big\}. \tag{10.41}$$

These conditions guarantee that $\Gamma_{\theta,b}(x, \pm N) \subseteq \mathcal{C}_{r,c}$ for each $x \in \mathcal{C}_{\lambda r,c} \cap \partial\Omega$ so, as far as the first inclusion in (10.40) is concerned, it suffices to show that

$$x', y' \in H, \;\; s \in \mathbb{R} \;\text{ so that }\; y' + sN \in \Gamma_{\theta,b}\big(x' + \varphi(x')N, N\big) \implies s > \varphi(y'). \tag{10.42}$$

Fix x', y', s as in the left-hand side of (10.42). Then

$$\cos(\theta/2)\sqrt{|y' - x'|^2 + (s - \varphi(x'))^2} < s - \varphi(x'). \tag{10.43}$$

Consequently, $s = s - \varphi(x') + \varphi(x') > \cos(\frac{\theta}{2})(|y' - x'|^2 + (s - \varphi(x'))^2)^{\frac{1}{2}} + (\varphi(x') - \varphi(y')) + \varphi(y')$. So to prove that $s > \varphi(y')$ it is enough to show that $\cos(\frac{\theta}{2})(|y' - x'|^2 + (s - \varphi(x'))^2)^{\frac{1}{2}} + (\varphi(x') - \varphi(y')) \geq 0$. This is trivially true if $y' = x'$, so there remains to consider the situation when $x' \neq y'$. Assuming that this is the case, define $A := \dfrac{|s - \varphi(x')|^2}{|y' - x'|^2}$ and $B := \dfrac{\varphi(x') - \varphi(y')}{|x' - y'|}$, in which scenario we must show that $\cos(\frac{\theta}{2})(1+A)^{\frac{1}{2}}+B\geq 0$. By construction, $A\geq 0$ and $B\in[-M,M]$, so it suffices to prove that

$$\cos(\tfrac{\theta}{2})(1+A)^{\frac{1}{2}}\geq M. \tag{10.44}$$

As a preamble, observe that $\cos(\frac{\theta}{2})(|y'-x'|^2+(s-\varphi(x'))^2)^{\frac{1}{2}}<s-\varphi(x')$ entails $\cos(\frac{\theta}{2})(1+A)^{\frac{1}{2}}<A^{\frac{1}{2}}$, or $\cos^2(\frac{\theta}{2})(1+A)<A$. Thus, $\dfrac{\cos^2(\frac{\theta}{2})}{1-\cos^2(\frac{\theta}{2})}<A$ and, further, $A>\cot^2(\frac{\theta}{2})$. Using this this lower bound on A in (10.44) yields $\cos(\frac{\theta}{2})(1+A)^{\frac{1}{2}}>\cos(\frac{\theta}{2})(1+\cot^2(\frac{\theta}{2}))^{\frac{1}{2}}=\cot^2(\frac{\theta}{2})$. Now $\cot^2(\frac{\theta}{2})\geq M$ if and only if $\tan^2(\frac{\theta}{2})\leq\frac{1}{M}$, which is true by our original choice of θ. This completes the proof of (10.42) and finishes the proof of the first inclusion in (10.40). The second inclusion in (10.40) is established in a similar fashion, completing the proof of the lemma. □

Our next result shows that suitable rotations of graphs of differentiable functions continue to be graphs of functions (enjoying the same degree of regularity as the original ones). This is going to be useful later, in the Proof of Theorem 12.3.

Lemma 10.6. *Assume that $\mathcal{O}\subseteq\mathbb{R}^{n-1}$ is an open neighborhood of the origin and $\varphi:\mathbb{R}^{n-1}\to\mathbb{R}$ is a function satisfying $\varphi(0')=0$, which is differentiable and whose derivative is continuous at $0'\in\mathbb{R}^{n-1}$. Let $\mathcal{R}$ be a rotation about the origin in $\mathbb{R}^n$ with the property that*

$$\mathcal{R}\ \text{ maps the vector }\ \frac{(\nabla\varphi(0'),-1)}{\sqrt{1+|\nabla\varphi(0')|^2}}\ \text{ into }\ -\mathbf{e}_n\in\mathbb{R}^n. \tag{10.45}$$

Then there exists a continuous, real-valued function ψ defined in a small neighborhood of $0'\in\mathbb{R}^{n-1}$ with the property that $\psi(0')=0$ and whose graph coincides, in a small neighborhood of $0\in\mathbb{R}^n$, with the graph of φ rotated by $\mathcal{R}$.

Furthermore, φ is of class $\mathscr{C}^{1,\alpha}$, for some $\alpha\in(0,1]$, if and only if so is ψ.

Proof. Matching the graph of φ, after being rotated by $\mathcal{R}$, by that of a function ψ comes down to ensuring that ψ is such that $\mathcal{R}(x', \varphi(x')) = (y', \psi(y'))$ can be solved both for x' in terms y', as well as for y' in terms x', near the origin in $\mathbb{R}^{n-1}$ in each instance. Let $\pi' : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the coordinate projection map of $\mathbb{R}^n$ onto the first $n-1$ coordinates, and denote by $\pi_n : \mathbb{R}^n \to \mathbb{R}$ the coordinate projection map of $\mathbb{R}^n$ onto the last coordinate. Then,

$$\begin{aligned}(y', y_n) = \mathcal{R}(x', \varphi(x')) \quad &\Leftrightarrow \quad \mathcal{R}^{-1}(y', y_n) = (x', \varphi(x')) \\ &\Leftrightarrow \quad \pi'\mathcal{R}^{-1}(y', y_n) = x' \;\text{ and }\; \pi_n\mathcal{R}^{-1}(y', y_n) = \varphi(x') \\ &\Leftrightarrow \quad F(y', y_n) = 0 \;\text{ and }\; x' = \pi'\mathcal{R}^{-1}(y', y_n), \end{aligned} \tag{10.46}$$

where F is the real-valued function defined in a neighborhood of the origin in $\mathbb{R}^n$ by

$$F(y', y_n) := \varphi(\pi'\mathcal{R}^{-1}(y', y_n)) - \pi_n\mathcal{R}^{-1}(y', y_n). \tag{10.47}$$

Then a direct calculation shows that $F(0', 0) = 0$ and

$$\begin{aligned}\partial_n F(y', y_n) &= \sum_{j=1}^{n-1} (\partial_j\varphi)(\pi'\mathcal{R}^{-1}(y', y_n))(\mathcal{R}^{-1}\mathbf{e}_n)\cdot\mathbf{e}_j - (\mathcal{R}^{-1}\mathbf{e}_n)\cdot\mathbf{e}_n \\ &= (\mathcal{R}^{-1}\mathbf{e}_n)\cdot((\nabla\varphi)(\pi'\mathcal{R}^{-1}(y', y_n))\,,\,-1) \\ &= \mathbf{e}_n\cdot\mathcal{R}((\nabla\varphi)(\pi'\mathcal{R}^{-1}(y', y_n))\,,\,-1). \end{aligned} \tag{10.48}$$

In particular, by (10.45),

$$\partial_n F(0', 0) = -\sqrt{1 + |\nabla\varphi(0')|^2} \neq 0. \tag{10.49}$$

Thus, by the Implicit Function Theorem, there exists a continuous real-valued function ψ defined in a small neighborhood of $0' \in \mathbb{R}^{n-1}$ such that $\psi(0') = 0$ and for which

$$F(y', y_n) = 0 \Longleftrightarrow y_n = \psi(y') \;\text{ whenever } (y', y_n) \text{ is near } 0. \tag{10.50}$$

From this and (10.46), all desired conclusions follow. $\square$

Chapter 11

Characterization of Lipschitz Domains in Terms of Cones

The main goal in this chapter is to discuss several types of cones conditions which fully characterize the class of Lipschitz domains in $\mathbb{R}^n$. The results presented here build on and generalize those from § 2 in [40]. To help put matters in the proper perspective, it is worth recalling that an open set $\Omega \subseteq \mathbb{R}^n$ with compact boundary and the property that there exists an open, circular, truncated, one-component cone Γ with vertex at $0 \in \mathbb{R}^n$ such that for every $x_0 \in \partial\Omega$ there exist $r > 0$ and a rotation $\mathcal{R}$ about the origin such that

$$x + \mathcal{R}(\Gamma) \subseteq \Omega, \qquad \forall\, x \in B(x_0, r) \cap \overline{\Omega} \tag{11.1}$$

is necessarily Lipschitz (the converse is also true). See Theorem 1.2.2.2 on p. 12 in [34] for a proof.

A different type of condition which characterizes Lipschitzianity has been recently discovered in [40]. This involves the notion of a transversal vector field to the boundary of a domain $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter which we now record. As a preamble, we remind the reader that $\partial^*\Omega$ denotes the reduced boundary of Ω and that $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff (outer-)measure in $\mathbb{R}^n$.

Definition 11.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set of locally finite perimeter, with outward unit normal ν, and fix a point $x_0 \in \partial\Omega$. Then, it is said that Ω has a* `continuous transversal vector field` `near` x_0 *provided there exists a continuous vector field* h *which is uniformly (outwardly) transverse to* $\partial\Omega$ *near* x_0, *in the sense that there exist* $r > 0$, $\kappa > 0$ *so that* $h : \partial\Omega \cap B(x_0, r) \to \mathbb{R}^n$ *is continuous and*

$$\nu \cdot h \geq \kappa \quad \mathcal{H}^{n-1}\text{-a.e. on} \quad B(x_0, r) \cap \partial^*\Omega. \tag{11.2}$$

Here is the statement of the result proved in [40] alluded to above.

Theorem 11.1. *Assume that* Ω *is a nonempty, proper open subset of* $\mathbb{R}^n$ *which has locally finite perimeter, and fix* $x_0 \in \partial\Omega$. *Then* Ω *is a Lipschitz domain near* x_0 *if and only if it has a continuous transversal vector field near* x_0 *and there exists* $r > 0$ *such that*

$$\partial(\Omega \cap B(x_0, r)) = \partial(\overline{\Omega \cap B(x_0, r)}). \tag{11.3}$$

We momentarily digress for the purpose of discussing an elementary result of topological nature which is going to be used shortly.

Lemma 11.2. *Let* E_1, E_2 *be two subsets of* $\mathbb{R}^n$ *with the property that*

$$\big(\partial E_1 \setminus \partial(\overline{E_1})\big) \cap \overline{E_2} = \varnothing \quad \text{and} \quad \big(\partial E_2 \setminus \partial(\overline{E_2})\big) \cap \overline{E_1} = \varnothing. \tag{11.4}$$

Then

$$\partial(E_1 \cap E_2) = \partial(\overline{E_1 \cap E_2}). \tag{11.5}$$

Proof. Since $\partial(\overline{E}) \subseteq \partial E$ for any set $E \subseteq \mathbb{R}^n$, the right-to-left inclusion in (11.5) always holds, so there remains to show that, granted (11.4), one has

$$\partial(E_1 \cap E_2) \subseteq \partial(\overline{E_1 \cap E_2}). \tag{11.6}$$

To this end, recall that

$$\partial(A \cap B) \subseteq (\overline{A} \cap \partial B) \cup (\partial A \cap \overline{B}), \qquad \forall\, A, B \subseteq \mathbb{R}^n, \tag{11.7}$$

which further implies

$$\partial(A\cap B)=\Big(\partial(A\cap B)\cap\overline{A}\cap\partial B\Big)\cup\Big(\partial(A\cap B)\cap\overline{B}\cap\partial A\Big). \tag{11.8}$$

From this and simple symmetry considerations we see that (11.6) will follow as soon as we check the validity of the inclusion

$$\partial(E_1\cap E_2)\cap(\overline{E_1}\cap\partial E_2)\subseteq\partial(\overline{E_1\cap E_2}). \tag{11.9}$$

To this end, we reason by contradiction and assume that there exist a point x and a number $r>0$ satisfying

$$\begin{gathered} x\in\partial(E_1\cap E_2),\qquad x\in\partial E_2,\quad\text{ and}\\ \text{either }\ B(x,r)\cap(\overline{E_1\cap E_2})=\varnothing,\ \text{ or }\ B(x,r)\subseteq\overline{E_1\cap E_2}. \end{gathered} \tag{11.10}$$

Note that if $B(x,r)\cap(\overline{E_1\cap E_2})=\varnothing$ then also $B(x,r)\cap(E_1\cap E_2)=\varnothing$, contradicting the fact that $x\in\partial(E_1\cap E_2)$. Thus, necessarily, $B(x,r)\subseteq\overline{E_1\cap E_2}$. However, this entails

$$x\in(\overline{E_1\cap E_2})^\circ\cap\partial E_2\subseteq\overline{E_1}\cap(\overline{E_2})^\circ\cap\partial E_2=\overline{E_1}\cap(\partial E_2\setminus\partial(\overline{E_2}))=\varnothing, \tag{11.11}$$

by (11.4). This shows that the conditions listed in (11.10) are contradictory and, hence, proves (11.9). $\square$

Definition 11.2. *A proper, nonempty open Ω subset of $\mathbb{R}^n$ is said to satisfy an* `exterior, uniform, continuously varying cone condition near a point` *$x_0\in\partial\Omega$ provided there exist two numbers $r,b>0$, an angle $\theta\in(0,\pi)$, and a function $h:B(x_0,r)\cap\partial\Omega\to S^{n-1}$ which is continuous at x_0 and such that*

$$\Gamma_{\theta,b}(x,h(x))\subseteq\mathbb{R}^n\setminus\Omega,\qquad\forall\,x\in B(x_0,r)\cap\partial\Omega. \tag{11.12}$$

Also, a nonempty, open set $\Omega\subseteq\mathbb{R}^n$ is said to satisfy a `global, exterior, uniform, continuously varying cone condition` *if Ω satisfies an interior uniform continuously varying cone condition near each point on $\partial\Omega$.*

Finally, define an `interior uniform continuously varying cone condition` *(near a boundary point, or globally) in an analogous manner, replacing* $\mathbb{R}^n \setminus \Omega$ *by* Ω *in* (11.12).

The global, interior, uniform, continuously varying cone condition has earlier appeared in [72] where N.S. Nadirashvili has used it as the main background geometrical hypothesis for the class of domains in which he proves a uniqueness theorem for the oblique derivative boundary value problem (cf. [72, Theorem 1, p. 327]). We shall revisit the latter topic in § 4. For now, our goal is to establish the following proposition, refining a result of similar flavor proved in [40][1].

Proposition 11.3. *Assume that* Ω *is a proper, nonempty open subset of* $\mathbb{R}^n$ *and that* $x_0 \in \partial\Omega$*. Then* Ω *is a Lipschitz domain near* x_0 *if and only if* Ω *satisfies an exterior, uniform, continuously varying cone condition near* x_0*.*

Proof. In one direction, if Ω is a Lipschitz domain near x_0 then the existence of $r, b > 0$, $\theta \in (0, \pi)$ and a function $h : B(x_0, r) \cap \partial\Omega \to S^{n-1}$ which is actually constant and such that (11.12) holds, follows from Lemma 10.5. The crux of the matter is, of course, dealing with the converse implication. In doing so, we shall employ the notation introduced in Definition 11.2. We begin by observing that condition (11.12) forces $B(x_0, r) \cap \partial\Omega \subseteq \overline{[(\Omega^c)^\circ]}$. In concert with the formula $(\Omega^c)^\circ = (\overline{\Omega})^c$, this yields $B(x_0, r) \cap \partial\Omega \subseteq \overline{[(\overline{\Omega})^c]}$. Hence, $B(x_0, r) \cap \partial\Omega \subseteq \overline{\Omega} \cap \overline{[(\overline{\Omega})^c]} = \partial(\overline{\Omega})$ and, further, we have $B(x_0, r) \cap \partial\Omega \subseteq B(x_0, r) \cap \partial(\overline{\Omega})$. Since the opposite inclusion is always true, we may ultimately deduce that

$$B(x_0, r) \cap \partial\Omega = B(x_0, r) \cap \partial(\overline{\Omega}). \tag{11.13}$$

[1] In the process, we also use the opportunity to correct a minor gap in the treatment in [40].

As a consequence of (11.13), we obtain

$$
\begin{aligned}
B(x_0,r)\cap(\overline{\Omega})^\circ &= B(x_0,r)\cap\big(\overline{\Omega}\setminus\partial(\overline{\Omega})\big)=\big(B(x_0,r)\cap\overline{\Omega}\big)\setminus\big(B(x_0,r)\cap\partial(\overline{\Omega})\big)\\
&= \big(B(x_0,r)\cap\overline{\Omega}\big)\setminus\big(B(x_0,r)\cap\partial\Omega\big)=B(x_0,r)\cap\big(\overline{\Omega}\setminus\partial\Omega\big)\\
&= B(x_0,r)\cap\Omega, \qquad (11.14)
\end{aligned}
$$

hence

$$
\Omega\cap B(x_0,r)=(\overline{\Omega})^\circ\cap B(x_0,r). \qquad (11.15)
$$

Next, fix $b_0\in(0,b)$ along with $\varepsilon\in(0,1-\cos(\theta/2))$. Then there exists $\theta_0\in(0,\theta)$ with the property that

$$
\cos(\theta_0/2)-\varepsilon>\cos(\theta/2) \quad \text{and} \quad \frac{b_0}{\cos(\theta_0/2)}<b. \qquad (11.16)
$$

Next, with $\varepsilon>0$ as above, select $r_0\in(0,r)$ such that

$$
|h(x)-h(x_0)|<\varepsilon \quad \text{whenever } \ x\in B(x_0,r_0)\cap\partial\Omega. \qquad (11.17)
$$

That this is possible is ensured by the continuity of the function h at x_0. We then claim that

$$
\Gamma_{\theta_0,b_0}(x,h(x_0))\subseteq\Gamma_{\theta,b}(x,h(x)), \qquad \forall\, x\in B(x_0,r_0)\cap\partial\Omega. \qquad (11.18)
$$

Indeed, if $x\in B(x_0,r_0)\cap\partial\Omega$ and $y\in\Gamma_{\theta_0,b_0}(x,h(x_0))$ then

$$
\begin{aligned}
(y-x)\cdot h(x) &= (y-x)\cdot h(x_0)+(y-x)\cdot(h(x)-h(x_0))\\
&> \cos(\theta_0/2)|y-x|-\varepsilon|y-x|=(\cos(\theta_0/2)-\varepsilon)|y-x|\\
&> \cos(\theta/2)|y-x|, \qquad (11.19)
\end{aligned}
$$

by the Cauchy-Schwarz inequality, the first inequality in (11.16) and condition (11.17). In addition, since $y \in \Gamma_{\theta_0,b_0}(x, h(x_0))$ forces $|y-x| < (\cos(\theta_0/2))^{-1}b_0$, it follows that

$$(y-x)\cdot h(x) \leq |y-x| < \frac{b_0}{\cos(\theta_0/2)} < b, \tag{11.20}$$

by the Cauchy-Schwarz inequality and the second inequality in (11.16). All together, this analysis proves (11.18). With this in hand, we deduce from (11.12) that

$$\Gamma_{\theta_0,b_0}(x, h(x_0)) \subseteq \mathbb{R}^n \setminus \Omega, \qquad \forall\, x \in B(x_0, r_0) \cap \partial\Omega. \tag{11.21}$$

Moving on, consider the open, proper, subset of $\mathbb{R}^n$ given by

$$D := (\Omega^c)^\circ \cap B(x_0, r_0). \tag{11.22}$$

Since, by (11.12), $\Gamma_{\theta,b}(x_0, h(x_0)) \subseteq (\Omega^c)^\circ$, it follows that D is also nonempty. The first claim we make about the set D is that

$$\partial D = \partial(\overline{D}). \tag{11.23}$$

To justify this, observe that $D = (\overline{\Omega})^c \cap B(x_0, r_0)$ and note that since

$$\partial E \setminus \partial(\overline{E}) = \partial E \cap (\overline{E})^\circ, \qquad \forall\, E \subseteq \mathbb{R}^n, \tag{11.24}$$

we have

$$\begin{aligned} \partial((\overline{\Omega})^c) \setminus \partial(\overline{(\overline{\Omega})^c}) = \partial(\overline{\Omega}) \cap \big(\overline{(\overline{\Omega})^c}\big)^\circ &= \partial(\overline{\Omega}) \cap \big(((\overline{\Omega})^\circ)\big)^c \\ &\subseteq \partial(\overline{\Omega}) \cap \big(\overline{\Omega}\big)^c = \varnothing. \end{aligned} \tag{11.25}$$

Having established this, (11.23) follows from Lemma 11.2.

Going further, the second claim we make about the set D introduced in (11.22) is that

$$\partial D \subseteq \big(\partial\Omega \cap \overline{B(x_0, r_0)}\big) \cup \partial B(x_0, r_0). \tag{11.26}$$

To see this, with the help of (11.7) we write

$$\begin{aligned}\partial D &\subseteq \big(\partial\big((\Omega^c)^\circ\big)\cap\overline{B(x_0,r_0)}\big)\cup\partial B(x_0,r_0)=\big(\partial\big((\overline{\Omega})^c\big)\cap\overline{B(x_0,r_0)}\big)\cup\partial B(x_0,r_0)\\ &= \big(\partial(\overline{\Omega})\cap\overline{B(x_0,r_0)}\big)\cup\partial B(x_0,r_0)=\big(\partial\Omega\cap\overline{B(x_0,r_0)}\big)\cup\partial B(x_0,r_0),\end{aligned}\tag{11.27}$$

where the last equality is a consequence of (11.13). This proves (11.26). Let us note here that, as a consequence of this, (11.21) and elementary geometrical considerations, we have

$$\eta:=\min\big\{b_0,\,\cos(\theta_0/2)\,r_0/2\big\}\Longrightarrow\Gamma_{\theta_0,\eta}(x,h(x_0))\subseteq D\quad\forall\,x\in B(x_0,r_0/2)\cap\partial D.\tag{11.28}$$

The third claim we make about the set D from (11.22) is that

$$\mathcal{H}^{n-1}(\partial D)<+\infty.\tag{11.29}$$

Of course, given (11.26), it suffices to show that there exists a finite constant $C=C(\theta,b)>0$ with the property that

$$\mathcal{H}^{n-1}(\partial\Omega\cap B(x_0,r_0))\leq Cr_0^{n-1}.\tag{11.30}$$

With this goal in mind, recall first that, in general, $\mathcal{H}^{n-1}(E)\leq C_n\lim\limits_{\delta\to0^+}\mathcal{H}^{n-1}_\delta(E)$, where $\mathcal{H}^{n-1}_\delta(E)$ denotes the infimum of all sums $\sum_{B\in\mathcal{B}}(\text{radius}\,B)^{n-1}$, associated with all covers $\mathcal{B}$ of E with open balls B of radii $\leq\delta$. Next, abbreviate $\Gamma:=\Gamma_{\theta_0,b_0}(0,h(x_0))$ so that (11.21) reads $x+\Gamma\subseteq\Omega^c$ for every $x\in B(x_0,r_0)\cap\partial\Omega$. Denote by L the one-dimensional space spanned by the vector $h(x_0)$ in $\mathbb{R}^n$. For some fixed $\lambda\in(0,1)$, to be specified later, consider $\Gamma_\lambda\subseteq\Gamma$ to be the open, truncated, circular, one-component cone of aperture $\lambda\,\theta_0$ with vertex at $0\in\mathbb{R}^n$ and having the same height b_0 and symmetry axis L as Γ. Elementary geometry gives

$$|x-y|<h,\ x\notin y+\Gamma,\ y\notin x+\Gamma\Longrightarrow|x-y|\leq\frac{\operatorname{dist}(x+L\,,\,y+L)}{\sin(\theta_0/2)}.\tag{11.31}$$

In subsequent considerations, it can be assumed that r_0 is smaller than a fixed fraction of b_0. To fix ideas, suppose henceforth that $0 < r_0 \leq b_0/10$.

In order to continue, select a small number $\delta \in (0, r_0)$ and cover $\partial\Omega \cap B(x_0, r_0)$ by a family of balls $\{B(x_j, r_j)\}_{j\in J}$ with $x_j \in \partial\Omega$, $0 < r_j \leq \delta$, for each $j \in J$. By Vitali's lemma, there is no loss of generality in assuming that $\{B(x_j, r_j/5)\}_{j\in J}$ are mutually disjoint. Then we have $\mathcal{H}^{n-1}_{\delta}(\partial\Omega \cap B(x_0, r_0)) \leq C_n \sum_{j\in J} r_j^{n-1}$. Let π be a fixed $(n-1)$-plane perpendicular to the axis of Γ and denote by A_j the projection of $(x_j + \Gamma_\lambda) \cap B(x_j, r_j/5)$ onto π. Clearly, we have $\mathcal{H}^{n-1}(A_j) \approx r_j^{n-1}$, for every $j \in J$, and there exists an $(n-1)$-dimensional ball of radius $3r$ in π containing all A_j's.

We now claim that $\lambda > 0$ can be chosen small as to ensure that the A_j's are mutually disjoint. Indeed, if $A_{j_1} \cap A_{j_2} \neq \varnothing$, for some $j_1, j_2 \in J$, then $\mathrm{dist}\,(x_{j_1} + L\,,\, x_{j_2} + L) \leq (r_{j_1} + r_{j_2})\,\sin(\lambda\,\theta_0/2)$. Moreover, we have $|x_{j_1} - x_{j_2}| \geq (r_{j_1} + r_{j_2})/5$, as $B(x_{j_1}, r_{j_1}/5) \cap B(x_{j_2}, r_{j_2}/5)$ is empty. Note that $|x_{j_1} - x_{j_2}| \leq 4r < b_0$. Since also $x_{j_1} \in \partial\Omega$ and $x_{j_1} \notin x_{j_2} + \Gamma \subseteq (\Omega^c)^\circ$ plus a similar condition with the roles of j_1 and j_2 reversed, it follows from (11.31) that $(r_{j_1} + r_{j_2})/5 \leq (r_{j_1} + r_{j_2})\sin(\lambda\,\theta_0/2)/\sin(\theta_0/2)$, or $\sin(\theta_0/2) < 5\,\sin(\lambda\,\theta_0/2)$. Taking $\lambda \in (0,1)$ sufficiently small, this leads to a contradiction. This finishes the proof of the claim that the A_j's are mutually disjoint if λ is small enough. Assuming that this is the case, we obtain $\sum_{j\in J} r_j^{n-1} \leq C \sum_{j\in J} \mathcal{H}^{n-1}(A_j) \leq C\mathcal{H}^{n-1}(\cup_{j\in J} A_j) \leq C r_0^{n-1}$, given the containment condition on the A_j's. As a consequence, $\mathcal{H}^{n-1}_{\delta}(\partial\Omega \cap B(x_0, r_0)) \leq C r_0^{n-1}$, so by taking the supremum over $\delta > 0$ we arrive at $\mathcal{H}^{n-1}(\partial\Omega \cap B(x_0, r_0)) \leq C r_0^{n-1}$. This finishes the proof of (11.30) and, hence, (11.29) holds.

In summary, the above analysis shows that D is a proper, nonempty open subset of $\mathbb{R}^n$, of finite perimeter and such that (11.28) holds. Granted this, if follows from Lemma 9.1 that if ν_D is the geometric measure theoretic outer unit normal to D then

$$\nu_D(x) \in \Gamma_{\pi-\theta_0,\eta}(0, h(x_0)) \quad \text{for each } x \in B(x_0, r_0/2) \cap \partial^* D. \tag{11.32}$$

Hence, the vector $h(x_0) \in S^{n-1}$ is transversal to ∂D near x_0 in the precise sense that

$$\nu_D(x) \cdot h(x_0) \geq \cos((\pi - \theta_0)/2) > 0 \quad \text{for each } x \in B(x_0, r_0/2) \cap \partial^* D. \tag{11.33}$$

From (11.23) (cf. also Lemma 11.2) and (11.33) we deduce that D is a Lipschitz domain near x_0.

The end-game in the proof of the proposition is as follows. Since D is a Lipschitz domain near x_0, it follows that $(\bar{D})^c$ is also a Lipschitz domain near x_0. In turn, this and the fact that, thanks to (11.15), we have $\Omega \cap B(x_0, r_0/2) = (\bar{D})^c \cap B(x_0, r_0/2)$, we may finally conclude that Ω is a Lipschitz domain near the point x_0. $\square$

Proposition 11.4. *Assume that $\Omega \subseteq \mathbb{R}^n$ is a nonempty open set which is not dense in $\mathbb{R}^n$, and suppose that $x_0 \in \partial\Omega$. Then Ω is a Lipschitz domain near x_0 if and only if there exist two numbers $r, b > 0$, an angle $\theta \in (0, \pi)$, and a function $h : B(x_0, r) \cap \partial\Omega \to S^{n-1}$ which is continuous at x_0 and such that*

$$B(x_0, r) \cap \partial\Omega = B(x_0, r) \cap \partial(\overline{\Omega}) \quad \textit{and} \tag{11.34}$$

$$\Gamma_{\theta,b}(x, h(x)) \subseteq \Omega, \quad \forall\, x \in B(x_0, r) \cap \partial\Omega. \tag{11.35}$$

Proof. This follows from applying Proposition 11.3 to the open, nonempty, proper subset $(\overline{\Omega})^c$ of $\mathbb{R}^n$ and keeping in mind (10.7). $\square$

It is instructive to observe that there is a weaker version of Propositions 11.3-11.4 (same conclusion, yet stronger hypotheses) but whose proof makes no use of results or tools from geometric measure theory. This is presented next.

Proposition 11.5. *Assume that $\Omega \subseteq \mathbb{R}^n$ is a nonempty, proper, open set. Then Ω is Lipschitz near $x_* \in \partial\Omega$ if and only if there exist $b, r > 0$, $\theta \in (0, \pi)$ and a function $h : B(x_*, r) \cap \partial\Omega \to S^{n-1}$ which* *is continuous at x_* and with the property that*

$$\Gamma_{\theta,b}(x,h(x)) \subseteq \Omega \quad \text{and} \quad \Gamma_{\theta,b}(x,-h(x)) \subseteq \mathbb{R}^n \setminus \Omega, \quad \forall\, x \in B(x_*,r) \cap \partial\Omega. \tag{11.36}$$

Proof. Assume first that the nonempty, proper, open set $\Omega \subseteq \mathbb{R}^n$ and the point $x_* \in \partial\Omega$ are such that (11.36) holds. Thanks to the analysis in (11.16)-(11.18), there is no loss of generality in assuming that the function $h : B(x_*,r) \cap \partial\Omega \to S^{n-1}$ is constant, say $h(x) \equiv v \in S^{n-1}$ for each $x \in B(x_*,r) \cap \partial\Omega$. Furthermore, since for any rotation $\mathcal{R} : \mathbb{R}^n \to \mathbb{R}^n$ we have

$$\mathcal{R}(\Gamma_{\theta,b}(x,\pm v)) = \Gamma_{\theta,b}(\mathcal{R}(x), \pm\mathcal{R}(v)), \tag{11.37}$$

there is no loss of generality in assuming that $v = \mathbf{e}_n$. Finally, performing a suitable translation, we can assume that $x_* = 0 \in \mathbb{R}^n$. Granted these, fix some small positive number c, say,

$$0 < c < \min\Big\{b\,\cos(\theta/2)\,,\, \frac{r}{\sqrt{1+(\cos(\theta/2))^2}}\Big\}, \tag{11.38}$$

and consider the cylinder

$$\mathcal{C} := B_{n-1}\big(0', c\,\cos(\theta/2)\big) \times (-c,c) \subseteq \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n. \tag{11.39}$$

Then the top lid of $\mathcal{C}$ is contained in $\Gamma_{\theta,b}(0,v) \subseteq \Omega$, whereas the bottom lid of $\mathcal{C}$ is contained in $\Gamma_{\theta,b}(0,-v) \subseteq (\mathbb{R}^n \setminus \Omega)^\circ = (\Omega^c)^\circ = (\overline{\Omega})^c$. We now make the claim that for every $x' \in B_{n-1}(0', c\,\cos(\theta/2))$,

$$\text{the interior of the line segment } L(x') := [(x',c),(x',-c)] \text{ intersects } \partial\Omega. \tag{11.40}$$

Indeed, if $x' \in B_{n-1}(0', c\,\cos(\theta/2))$ is such that the (relative) interior of $L(x')$ is disjoint from $\partial\Omega$, the fact that $\mathbb{R}^n = \Omega \cup \partial\Omega \cup (\overline{\Omega})^c$ with the three sets appearing in the right-hand side mutually disjoint, implies that Ω and $(\overline{\Omega})^c$ form an open cover of $L(x')$. Since $L(x') \cap \Omega$ is nonempty (as it contains (x',c)), $L(x') \cap (\overline{\Omega})^c$ is nonempty (as it contains $(x',-c)$), and $\Omega \cap (\overline{\Omega})^c = \varnothing$, this contradicts the fact that $L(x')$ is connected. This proves that there exists $x_0 \in L(x')$ with the property that $x_0 \in \partial\Omega$. There remains to observe that, necessarily, x_0 is different from the endpoints of $L(x')$ in order to conclude that this point actually belongs to the (relative) interior of $L(x')$. This finishes the proof of (11.40).

Our next claim is that, in fact (with $\#E$ denoting the cardinality of the set E)

$$\#(L(x') \cap \partial\Omega) = 1, \qquad \forall\, x' \in B_{n-1}(0', c\,\cos(\theta/2)). \tag{11.41}$$

To justify this, let $x = (x', x_n) \in L(x') \cap \partial\Omega$. Then

$$|x| = \sqrt{|x'|^2 + x_n^2} \leq \sqrt{c^2(\cos(\theta/2))^2 + c^2} = c\sqrt{1 + (\cos(\theta/2))^2} < r, \tag{11.42}$$

so $x \in B(0, r) \cap \partial\Omega$. Consequently, from (11.36) and conventions,

$$\Gamma_{\theta,b}(x, \mathbf{e}_n) \subseteq \Omega \quad \text{ and } \quad \Gamma_{\theta,b}(x, -\mathbf{e}_n) \subseteq (\mathbb{R}^n \setminus \Omega)^\circ. \tag{11.43}$$

In turn, this forces (with $\mathcal{I}(y, z)$ denoting the relatively open line segment with endpoints $y, z \in \mathbb{R}^n$)

$$\mathcal{I}(x, x + b\,\mathbf{e}_n) \subseteq \Omega \quad \text{ and } \quad \mathcal{I}(x, x - b\,\mathbf{e}_n) \subseteq (\overline{\Omega})^c \tag{11.44}$$

and, hence, $\mathcal{I}(x - b\,\mathbf{e}_n, x + b\,\mathbf{e}_n) \cap \partial\Omega = \{x\}$. With this in hand, (11.41) follows after noticing that the (relative) interior of $L(x')$ is contained in $\mathcal{I}(x - b\,\mathbf{e}_n, x + b\,\mathbf{e}_n)$ since, by design, $c < b\,\cos(\theta/2) < b$.

Having established (11.41), it is then possible to define a function

$$\varphi : B_{n-1}(0', c\,\cos(\theta/2)) \longrightarrow (-c, c) \tag{11.45}$$

in an unambiguous fashion by setting, for every $x' \in B_{n-1}(0', c\,\cos(\theta/2))$,

$$\varphi(x') := x_n \quad \text{if} \quad (x', x_n) \in L(x') \cap \partial\Omega. \tag{11.46}$$

Then, by design (recall (11.39)), we have

$$\mathcal{C} \cap \partial\Omega = \{x = (x', x_n) \in \mathcal{C} : \, x_n = \varphi(x')\}, \tag{11.47}$$

and we now proceed to show that φ defined in (11.45)-(11.46) is a Lipschitz function. Concretely, if we now select two arbitrary points $x', y' \in B_{n-1}(0', c\,\cos(\theta/2))$, then $(y', \varphi(y'))$ belongs to $\partial\Omega$, therefore $(y', \varphi(y')) \notin \Gamma_{\theta,b}((x', \varphi(x')), \pm\mathbf{e}_n)$. This implies

$$
\begin{aligned}
\pm((y', \varphi(y')) - (x', \varphi(x'))) \cdot \mathbf{e}_n &\leq \cos(\theta/2)\,|(y', \varphi(y')) - (x', \varphi(x'))| \\
&\leq \cos(\theta/2)\,|y' - x'|. \qquad (11.48)
\end{aligned}
$$

Thus, ultimately, $|\varphi(y') - \varphi(x')| \leq \cos(\theta/2)\,|y' - x'|$, which shows that φ is a Lipschitz function, with Lipschitz constant $\leq \cos(\theta/2)$. Based on the classical result of E.J. McShane [68] and H. Whitney [91], the function (11.45) may be extended to the entire Euclidean space $\mathbb{R}^{n-1}$ to a Lipschitz function, with Lipschitz constant $\leq \cos(\theta/2)$.

Going further, since the cone condition (11.36) also entails that the point $x_0 \in \partial\Omega$ is the limit of points from $\Gamma_{\theta,b}(x_0, h(x_0)) \subseteq (\mathbb{R}^n \setminus \Omega)^\circ$, we may conclude that $x_0 \in \overline{(\bar{\Omega})^c}$, i.e., $x_0 \notin (\overline{\Omega})^\circ$. With this and (11.47) in hand, we may then invoke Proposition 10.1 in order to conclude that Ω is a Lipschitz domain near 0.

Finally, the converse implication in the statement of the proposition is a direct consequence of Lemma 10.5. $\square$

Definition 11.3. *Call a set $\Omega \subseteq \mathbb{R}^n$* `starlike with respect to` *$x_0 \in \Omega$ if $\mathcal{I}(x, x_0) \subseteq \Omega$ for all $x \in \Omega$, where $\mathcal{I}(x, x_0)$ denotes the open line segment in $\mathbb{R}^n$ with endpoints x and x_0.*

Also, call a set $\Omega \subseteq \mathbb{R}^n$ `starlike with respect to a ball` *$B \subseteq \Omega$ if $\mathcal{I}(x, y) \subseteq \Omega$ for all $x \in \Omega$ and $y \in B$ (that is, Ω is starlike with respect to any point in B).*

Theorem 11.6. *Let Ω be an open, proper, nonempty subset of $\mathbb{R}^n$. Then Ω is a locally Lipschitz domain if and only if every $x_* \in \partial\Omega$ has an open neighborhood $\mathcal{O} \subseteq \mathbb{R}^n$ with the property that $\Omega \cap \mathcal{O}$ is starlike with respect to some ball.*

In particular, any bounded convex domain is Lipschitz.

Proof. Pick an arbitrary point $x_* \in \partial\Omega$ and let $\mathcal{O} \subseteq \mathbb{R}^n$ be an open neighborhood of x_* with the property that $\Omega\cap\mathcal{O}$ is starlike with respect to a ball $B(x_0,r)\subseteq\Omega\cap\mathcal{O}$. For each $x\in\mathbb{R}^n\setminus\overline{B(x_0,r/2)}$, consider the circular cone with vertex at x and axis along x_0-x described as

$$C(x):=\Big\{y\in\mathbb{R}^n:\ \sqrt{1-\tfrac{r}{2|x-x_0|}}\,|y-x|<(y-x)\cdot\tfrac{x_0-x}{|x_0-x|}<\tfrac{|x_0-x|^2-(r/2)^2}{|x_0-x|}\Big\}. \tag{11.49}$$

Elementary geometry then shows that

$$C(x)\subseteq\bigcup_{y\in B(x_0,r/2)}\mathcal{I}(x,y)\subseteq\mathcal{O}\cap\Omega,\qquad\forall\,x\in(\mathcal{O}\cap\Omega)\setminus\overline{B(x_0,r/2)}, \tag{11.50}$$

where the second inclusion is a consequence of the fact that $\mathcal{O}\cap\Omega$ is starlike with respect to $B(x_0,r/2)$. Then, for each $x\in\mathcal{O}\cap\partial\Omega$, there exists a sequence $\{x_j\}_{j\in\mathbb{N}}$ of points in $\mathcal{O}\cap\Omega$ such that $x_j\to x$ as $j\to+\infty$, hence

$$C(x)\subseteq\bigcup_{j\in\mathbb{N}}C(x_j). \tag{11.51}$$

In concert with (11.50), this implies that

$$C(x)\subseteq\mathcal{O}\cap\Omega,\qquad\forall\,x\in\mathcal{O}\cap\partial\Omega. \tag{11.52}$$

Next, for each $b>0$ and $x\in\mathbb{R}^n\setminus\overline{B(x_0,r/2)}$ denote by $\widetilde{C}_b(x)$ the cone with vertex at x, same aperture as $C(x)$, axis pointing in the opposite direction to that of $C(x)$, and height b. We then claim that there exist $b>0$ and $\rho>0$ with the property that

$$\widetilde{C}_b(x)\subseteq\mathbb{R}^n\setminus\Omega,\qquad\forall\,x\in B(x_*,\rho)\cap\partial\Omega. \tag{11.53}$$

To justify this claim, note that since $\mathcal{O}$ is an open neighborhood of x_* it is possible to select $b,\rho>0$ sufficiently small so that

$$\widetilde{C}_b(x)\subseteq\mathcal{O}\qquad\forall\,x\in B(x_*,\rho). \tag{11.54}$$

Assuming that this is the case, the existence of a point $x \in B(x_*, \rho) \cap \partial\Omega$ for which there exists $\widehat{x} \in \widetilde{C}_b(x) \cap \Omega$ would entail, thanks to (11.54) and (11.50),

$$x \in C(\widehat{x}) \subseteq \mathcal{O} \cap \Omega, \tag{11.55}$$

which contradicts the fact that $x \in \partial\Omega$. This finishes the proof of the claim made in (11.53).

Having established (11.52) and (11.53), Proposition 11.5 applies and yields that Ω is Lipschitz near x_*. Since $x_* \in \partial\Omega$ has been arbitrarily chosen we may therefore conclude that Ω is locally Lipschitz. This establishes one of the implications in the equivalence formulated in the statement of the theorem.

In the opposite direction, observe that if $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant $M > 0$ and if $x_0' \in \mathbb{R}^{n-1}$ and $t > 0$ are given, then

$$\begin{array}{l}\text{the open segment with endpoints } (x_0', t + \varphi(x_0')) \text{ and } (x', \varphi(x')) \text{ belongs} \\ \text{to the (open) upper-graph of } \varphi \text{ whenever } x' \in \mathbb{R}^{n-1} \text{ satisfies } |x'| < t/M.\end{array} \tag{11.56}$$

Then the desired conclusion (i.e., that Ω is locally starlike in the sense explained in the statement of the theorem) follows from this and (10.1). $\square$

Chapter 12

Characterizing Lyapunov Domains in Terms of Pseudo-Balls

This chapter contains the main result in this paper of geometrical flavor, namely the geometric characterization of Lyapunov domains in terms of a uniform, two-sided pseudo-ball condition. To set the stage, we first make the following definition.

Definition 12.1. *Let E be an arbitrary, proper, nonempty, subset of $\mathbb{R}^n$.*

(i) The set E is said to satisfy an `interior pseudo-ball condition` *at $x_0 \in \partial E$ with shape function ω as in* (1.14) *provided there exist a, $b > 0$ and $h \in S^{n-1}$ such that $\mathscr{G}^{\omega}_{a,b}(x_0, h) \subseteq E$.*

(ii) The set E is said to satisfy an `exterior pseudo-ball condition` *at $x_0 \in \partial E$ with shape function ω as in* (1.14) *provided $E^c := \mathbb{R}^n \setminus E$ satisfies an interior pseudo-ball condition at the point x_0 with shape function ω.*

(iii) The set E is said to satisfy a `two-sided pseudo-ball condition` *at $x_0 \in \partial E$ with shape function ω as in* (1.14) *provided E satisfies both an interior and an exterior pseudo-ball condition at $x_0 \in \partial E$ with shape function ω.*

(iv) The set E is said to satisfy a `uniform hour-glass condition near` *$x_0 \in \partial E$ with shape function ω as in* (1.14) *provided there exists $r > 0$ such that E satisfies a two-sided pseudo-*

ball condition at each point $x \in B(x_0, r) \cap \partial E$ with shape function ω and truncation height independent of x.

(v) Finally, the set E is said to satisfy a `uniform hour-glass condition` *with shape function ω as in* (1.14) *provided both E and E^c satisfy a pseudo-ball condition at each point $x \in \partial E$ with shape function ω and height independent of x.*

While Definition 12.1 only requires that ω is as in (1.14), for the rest of this chapter we will also assume that ω satisfies (1.16), i.e., ω is as in (8.11).

That the terminology "hour-glass condition" employed above is justified is made transparent in the lemma below.

Lemma 12.1. *Let E be a subset of $\mathbb{R}^n$ which satisfies a two-sided pseudo-ball condition at point $x_0 \in \partial E$ with shape function ω as in* (8.11). *That is, there exist a, $b > 0$ and $h_\pm \in S^{n-1}$ such that $\mathscr{G}^{\omega}_{a,b}(x_0, h_+) \subseteq E$ and $\mathscr{G}^{\omega}_{a,b}(x_0, h_-) \subseteq E^c := \mathbb{R}^n \setminus E$. Then necessarily $h_+ = -h_-$.*

Proof. This is an immediate consequence of Corollary 8.4. □

Remarkably, if $E \subseteq \mathbb{R}^n$ satisfies a uniform hour-glass condition then the function $h : \partial E \to S^{n-1}$, assigning to each boundary point $x \in \partial E$ the direction $h(x) \in S^{n-1}$ of the pseudo-ball with apex at x contained in E, turns out to be continuous. A precise, local version of this result is recorded next.

Lemma 12.2. *Assume that the set $E \subseteq \mathbb{R}^n$ satisfies a uniform hour-glass condition near $x_* \in \partial E$ with shape function ω as in* (8.11)*, height $b > 0$ and aperture $a > 0$. Let $\varepsilon = \varepsilon(\omega, \eta, R, a, b) > 0$ be as in Lemma 8.3 and define*

$$\widehat{\omega} : [0,1] \to \big[0, \tfrac{\varepsilon R}{2}\big], \quad \widehat{\omega}(t) := \tfrac{\varepsilon}{2}\, \omega^{-1}\big(\omega(R)t\big)t, \quad \forall\, t \in [0,1]. \tag{12.1}$$

Since $\widehat{\omega}$ is continuous, increasing and bijective, it is meaningful to consider its inverse, i.e., the *function*

$$\widetilde{\omega} : \big[0, \tfrac{\varepsilon R}{2}\big] \to [0,1], \quad \widetilde{\omega}(t) := \widehat{\omega}^{\,-1}(t), \quad \forall\, t \in \big[0, \tfrac{\varepsilon R}{2}\big], \tag{12.2}$$

which is also continuous and increasing.

Then there exists a number $r > 0$ such that the function $h : B(x_*, r) \cap \partial E \to S^{n-1}$, defined at each point $x \in B(x_*, r) \cap \partial E$ by the demand that $h(x)$ is the unique vector in S^{n-1} with the property that $\mathscr{G}^{\omega}_{a,b}(x, h(x)) \subseteq E$, is well-defined and continuous. In fact, with $\widetilde{\omega}$ as in (12.2)*, one has*

$$h \in \mathscr{C}^{\,\widetilde{\omega}}\big(B(x_*, r) \cap \partial E\,,\, S^{n-1}\big). \tag{12.3}$$

Proof. Let $r > 0$, ω as in (8.11), and $a, b > 0$ be such that E satisfies a two-sided pseudo-ball condition at each point $x \in B(x_*, r) \cap \partial E$ with shape function ω, height b and aperture a. The fact that for each $x \in B(x_*, r) \cap \partial E$ there exists a unique vector $h(x) \in S^{n-1}$, which is unequivocally determined by the demand that $\mathscr{G}^{\omega}_{a,b}(x, h(x)) \subseteq E$, follows from our assumption on E and Lemma 12.1. Consequently, we also have $\mathscr{G}^{\omega}_{a,b}(x, -h(x)) \subseteq \mathbb{R}^n \setminus E$.

We are left with proving that the mapping $B(x_*, r) \cap \partial E \ni x \mapsto h(x) \in S^{n-1}$ is continuous and, in the process, estimate its modulus of continuity. With this goal in mind, pick two arbitrary points $x_0, x_1 \in B(x_*, r) \cap \partial E$. We then have $\mathscr{G}^{\omega}_{a,b}(x_0, h(x_0)) \cap \mathscr{G}^{\omega}_{a,b}(x_1, -h(x_1)) = \varnothing$ since the former set is contained in E and the latter set is contained in $\mathbb{R}^n \setminus E$. In turn, from this, Lemma 8.3, and (12.1) we infer that

$$|x_0 - x_1| \geq \tfrac{\varepsilon}{2}\, \omega^{-1}\Big(\omega(R)\tfrac{|h(x_0)-h(x_1)|}{2}\Big)\Big|\tfrac{h(x_0)-h(x_1)}{2}\Big| = \widehat{\omega}\Big(\tfrac{|h(x_0)-h(x_1)|}{2}\Big). \tag{12.4}$$

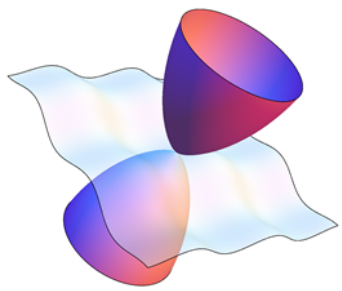
As a consequence, if $0 < r < \frac{\varepsilon R}{4}$ to begin with, we obtain from (12.4) and (12.2) that

$$|h(x_0) - h(x_1)| \leq 2\,\widetilde{\omega}(|x_0 - x_1|), \qquad \forall\, x_0, x_1 \in B(x_*, r) \cap \partial E. \tag{12.5}$$

This shows that $h \in \mathscr{C}^{\,\widetilde{\omega}}\big(B(x_*, r) \cap \partial E, S^{n-1}\big)$, as desired. $\square$

We are now in a position to formulate the main result in this chapter.

Theorem 12.3. *Let Ω be an open, proper, nonempty subset of $\mathbb{R}^n$ and assume that ω is as in* (8.11) *and $x_0 \in \partial\Omega$. Then Ω satisfies a uniform hour-glass condition with shape function ω near x_0 if and only if Ω is of class $\mathscr{C}^{1,\omega}$ near x_0.*



Let us momentarily pause to record an immediate consequence of Theorem 12.3 which is particularly useful in applications.

Corollary 12.4. *Given ω as in* (8.11)*, an open proper nonempty subset Ω of $\mathbb{R}^n$ with compact boundary is of class $\mathscr{C}^{1,\omega}$ if and only if Ω satisfies a uniform hour-glass condition with shape function ω.*

As a corollary, an open proper nonempty subset Ω of $\mathbb{R}^n$ with compact boundary is of class $\mathscr{C}^{1,1}$ if and only if it satisfies a uniform two-sided ball condition.

Proof. The first claim in the statement is a direct consequence of Theorem 12.3, while the last claim follows from the first with the help of part (*iii*) in Lemma 8.1. $\square$

One useful ingredient in the proof of Theorem 12.3, of independent interest, is the differentiability criterion of geometrical nature presented in the proposition below.

Proposition 12.5. *Assume that $U \subseteq \mathbb{R}^{n-1}$ is an arbitrary set, and that $x_* \in U^\circ$. Given a function $f : U \to \mathbb{R}$, denote by G_f the graph of f, i.e., $G_f := \{(x, f(x)) : x \in U\} \subseteq \mathbb{R}^n$.*

Then f is differentiable at the point x_* if and only if f is continuous at x_* and there exists a non-horizontal vector $N \in \mathbb{R}^n$ (i.e., satisfying $N \cdot \mathbf{e}_n \neq 0$) with the following significance. For every angle $\theta \in (0, \pi)$ there exists $\delta > 0$ with the property that $G_f \cap B((x_*, f(x_*)), \delta)$ lies in between the cones $\Gamma_{\theta,\delta}((x_*, f(x_*)), N)$ and $\Gamma_{\theta,\delta}((x_*, f(x_*)), -N)$, i.e.,

$$G_f \cap B((x_*, f(x_*)), \delta) \subseteq \mathbb{R}^n \setminus \Big[\Gamma_{\theta,\delta}((x_*, f(x_*)), N) \cup \Gamma_{\theta,\delta}((x_*, f(x_*)), -N)\Big]. \tag{12.6}$$

If this happens, then necessarily N is a scalar multiple of $(\nabla f(x_*), -1) \in \mathbb{R}^n$.

Proof. Assume that f is differentiable at x_*. Then f is continuous at x_*. To proceed, take

$$N := \frac{(\nabla f(x_*), -1)}{\sqrt{1 + |\nabla f(x_*)|^2}} \in \mathbb{R}^n. \tag{12.7}$$

Clearly, $|N| = 1$ and $N \cdot \mathbf{e}_n = -(1 + |\nabla f(x_*)|^2)^{-1/2} \neq 0$, so N is non-horizontal. Then, given $\theta \in (0, \pi)$, the fact that f is differentiable at x_* implies that there exists $\delta > 0$ for which

$$|f(x) - f(x_*) - (\nabla f(x_*)) \cdot (x - x_*)| < \cos(\theta/2)|x - x_*| \quad \forall\, x \in B(x_*, \delta) \cap U. \tag{12.8}$$

For any $x \in B(x_*, \delta) \cap U$ we may then estimate

$$\begin{aligned} \Big|\big((x, f(x)) - (x_*, f(x_*))\big) \cdot N\Big| &= \frac{|(\nabla f(x_*)) \cdot (x - x_*) - f(x) + f(x_*)|}{\sqrt{1 + |\nabla f(x_*)|^2}} \\ &\leq |(\nabla f(x_*)) \cdot (x - x_*) - f(x) + f(x_*)| < \cos(\theta/2)|x - x_*| \\ &< \cos(\theta/2)|(x, f(x)) - (x_*, f(x_*))|, \end{aligned} \tag{12.9}$$

which (recall that $|N| = 1$) shows that

$$x \in B(x_*, \delta) \cap U \Longrightarrow (x, f(x)) \notin \Gamma_{\theta,\delta}((x_*, f(x_*)), \pm N). \tag{12.10}$$

Upon observing that any point in $G_f \cap B((x_*, f(x_*)), \delta)$ is of the form $(x, f(x))$ for some point $x \in B(x_*, \delta) \cap U$, based on (12.10) we may conclude that (12.6) holds.

For the converse implication, suppose that f is continuous at x_* and assume that there exists a non-horizontal vector $N \in \mathbb{R}^n$ with the property that for every angle $\theta \in (0, \pi)$ there exists $\delta > 0$ such that (12.6) holds. By dividing N by the nonzero number $-N \cdot \mathbf{e}_n$, we may assume that the n-th component of N is -1 to begin with. That is, $N = (N', -1)$ for some $N' \in \mathbb{R}^{n-1}$.

Fix an arbitrary number $\varepsilon \in (0, 1/2)$ and pick an angle $\theta \in (0, \pi)$ sufficiently close to π so that $0 < \cos(\theta/2) < \varepsilon/\sqrt{1 + |N'|^2}$. Then, by assumption, there exists $\delta_0 > 0$ with the property that if $x \in U$ is such that $|(x, f(x)) - (x_*, f(x_*))| < \delta_0$ then $(x, f(x)) \notin \Gamma_{\theta,\delta}((x_*, f(x_*)), \pm N)$, i.e.,

$$\begin{aligned} \Big|((x, f(x)) - (x_*, f(x_*))) \cdot (N', -1)\Big| &\leq \cos(\theta/2)|(N', -1)||(x, f(x)) - (x_*, f(x_*))| \\ &\leq \varepsilon\sqrt{|x - x_*|^2 + (f(x) - f(x_*))^2} \leq \varepsilon\big[|x - x_*| + |f(x) - f(x_*)|\big]. \end{aligned} \tag{12.11}$$

In turn, this forces (recall that $0 < \varepsilon < \frac{1}{2}$)

$$\begin{aligned} |f(x) - f(x_*)| &\leq \big|((x, f(x)) - (x_*, f(x_*))) \cdot (N', -1)\big| + |(x - x_*) \cdot N'| \\ &\leq \varepsilon\big[|x - x_*| + |f(x) - f(x_*)|\big] + |x - x_*||N'| \\ &\leq (\tfrac{1}{2} + |N'|)|x - x_*| + \tfrac{1}{2}|f(x) - f(x_*)|. \end{aligned} \tag{12.12}$$

Absorbing the last term above in the left-most side of (12.12) yields

$$\tfrac{1}{2}|f(x) - f(x_*)| \leq (\tfrac{1}{2} + |N'|)|x - x_*|. \tag{12.13}$$

We have therefore proved that there exists $\delta_0 > 0$ for which

$$\begin{aligned} &x \in U \;\text{ and }\; |(x, f(x)) - (x_*, f(x_*))| < \delta_0 \\ &\qquad \Longrightarrow |f(x) - f(x_*)| \leq (1 + 2|N'|)|x - x_*|. \end{aligned} \tag{12.14}$$

Returning with this back in (12.11) then yields

$$\begin{aligned} &x \in U \;\text{ and }\; |(x, f(x)) - (x_*, f(x_*))| < \delta_0 \Longrightarrow \\ &\qquad \Big|((x, f(x)) - (x_*, f(x_*))) \cdot (N', -1)\Big| \leq 2\varepsilon(1 + |N'|)|x - x_*|. \end{aligned} \tag{12.15}$$

Since we are assuming that f is continuous at the point x_*, it follows that there exists $\delta_1 > 0$ with the property that

$$x \in U \ \text{ and } \ |x - x_*| < \delta_1 \Longrightarrow |f(x) - f(x_*)| < \delta_0/\sqrt{2}. \tag{12.16}$$

Introducing $\delta := \min\{\delta_1, \delta_0/\sqrt{2}\}$, implication (12.16) therefore guarantees that

$$x \in U \ \text{ and } \ |x - x_*| < \delta \Longrightarrow |(x, f(x)) - (x_*, f(x_*))| < \delta_0. \tag{12.17}$$

Consequently, from this and (12.15) we deduce that

$$x \in B(x_*, \delta) \cap U \Longrightarrow \left| \frac{((x, f(x)) - (x_*, f(x_*))) \cdot (N', -1)}{|x - x_*|} \right| \leq 2\varepsilon(1 + |N'|). \tag{12.18}$$

Since $\varepsilon \in (0, 1/2)$ was arbitrary, this translates into saying that

$$\lim_{x \to x_*,\, x \in U} \frac{f(x) - f(x_*) - N' \cdot (x - x_*)}{|x - x_*|} = 0. \tag{12.19}$$

This proves that f is differentiable at x_* and, in fact, $\nabla f(x_*) = N'$. Hence, in particular, N is a scalar multiple of $(N', -1) = (\nabla f(x_*), -1)$. The proof of the proposition is therefore finished. $\square$

We are now ready to discuss the

Proof of Theorem 12.3. In a first stage, assume that Ω is an open, proper, nonempty subset of $\mathbb{R}^n$ which satisfies a uniform hour-glass condition with shape function ω (as in (8.11)) near $x_* \in \partial\Omega$. In other words, there exist $b > 0$ and $r_* > 0$, along with a function $h : B(x_*, r_*) \cap \partial\Omega \to S^{n-1}$ such that

$$\mathscr{G}^{\omega}_{a,b}(x, h(x)) \subseteq \Omega \ \text{ and } \ \mathscr{G}^{\omega}_{a,b}(x, -h(x)) \subseteq \Omega^c \ \text{ for every } \ x \in B(x_*, r_*) \cap \partial\Omega. \tag{12.20}$$

Note that the uniform hour-glass condition, originally introduced in part (iv) of Definition 12.1, may be written as above thanks to Corollary 8.4. Going further, Lemma 12.2 then guarantees (by eventually decreasing $r_* > 0$ if necessary) that the function $h : B(x_*, r_*) \cap \partial E \to S^{n-1}$ belongs to

$\mathscr{C}^{\widetilde{\omega}}$, where $\widetilde{\omega}$ is as in (12.2). Hence, in particular, h is continuous at x_*. Having established this, from part (v) of Lemma 8.1 and Proposition 11.3 we then deduce that Ω is a Lipschitz domain near x_*. Hence, there exist an $(n-1)$-dimensional plane $H \subseteq \mathbb{R}^n$ passing through the point x_*, a choice of the unit normal N to H, a Lipschitz function $\varphi : H \to \mathbb{R}$ and a cylinder $\mathcal{C}_{r,c}$ such that (10.1)-(10.2) hold. Without loss of generality we may assume that x_* is the origin in $\mathbb{R}^n$, that H is the canonical horizontal $(n-1)$-dimensional plane $\mathbb{R}^{n-1} \times \{0\} \subseteq \mathbb{R}^n$ and that $N = \mathbf{e}_n$. In this setting, our goal is to show that

$$\text{the Lipschitz function } \ \varphi : \mathbb{R}^{n-1} \to \mathbb{R} \ \text{ is actually of class } \mathscr{C}^{1,\omega} \text{ near } 0' \in \mathbb{R}^{n-1}. \tag{12.21}$$

As a preamble, we shall show that

$$h(x) \cdot \mathbf{e}_n \neq 0 \ \text{ for every } \ x \in B(0, r_*) \cap \partial\Omega. \tag{12.22}$$

To prove (12.22), assume that there exists $x_0 \in B(0,r_*) \cap \partial\Omega$ such that $h(x_0) \cdot \mathbf{e}_n = 0$, with the goal of deriving a contradiction. Then, on the one hand, (12.20) gives that $\mathscr{G}^{\omega}_{a,b}(x_0, -h(x_0)) \subseteq \Omega^c$, whereas Lemma 10.5 guarantees that $\Gamma_{\theta_0,b_0}(x_0, \mathbf{e}_n) \subseteq \Omega$ if $\theta_0 := 2\arctan\left(\frac{1}{M}\right)$ and $b_0 > 0$ is sufficiently small, where M is the Lipschitz constant of the function φ. Given the locations of the aforementioned pseudo-ball and cone, the desired contradiction will follow as soon as we show that

$$\mathscr{G}^{\omega}_{a,b}(x_0, -h(x_0)) \cap \Gamma_{\theta_0,b_0}(x_0, \mathbf{e}_n) \neq \varnothing. \tag{12.23}$$

To this end, it suffices to look at the cross-section of $\mathscr{G}^{\omega}_{a,b}(x_0, -h(x_0))$ and $\Gamma_{\theta_0,b_0}(x_0, \mathbf{e}_n)$ with the two-dimensional plane π spanned by the orthogonal unit vectors $h(x_0)$ and $\mathbf{e}_n$. To fix ideas, choose a system of coordinates in π so that $\mathbf{e}_n$ is vertical and $-h(x_0)$ is horizontal, both pointing in the positive directions of these respective axes. In such a setting, it follows that there exists $m \in (0, +\infty)$ with the property that the cross-section of the truncated cone contains all points (x, y) with coordinates satisfying $y > mx$ for $x > 0$ sufficiently small. On the other hand, the portion of the boundary of the cross-section of the pseudo-ball lying in the first quadrant near the origin is described by the equation $\sqrt{x^2+y^2}\,\omega(\sqrt{x^2+y^2}) = x$. Hence, $\omega\big(x\sqrt{1+(y/x)^2}\big) = 1/\sqrt{1+(y/x)^2}$ and, given that $\omega(t) \searrow 0$ as $t \searrow 0$, this forces $y/x \to +\infty$ as $x \searrow 0$. From this, the desired conclusion follows, completing the proof of (12.22).

Moving on, based on (12.22), the fact that φ is continuous, part (v) of Lemma 8.1, and the geometric differentiability criterion presented in Proposition 12.5, we deduce that φ is differentiable at each point near $0' \in \mathbb{R}^{n-1}$ and, in addition,

$$h(x', \varphi(x')) \text{ is parallel to } (\nabla\varphi(x'), -1) \text{ for each } x' \text{ near } 0' \in \mathbb{R}^{n-1}. \tag{12.24}$$

We now make the claim that for each x' near $0' \in \mathbb{R}^{n-1}$ the vector $(\nabla\varphi(x'), -1)$ points away from Ω, in the sense that

$$(x', \varphi(x')) - t(\nabla\varphi(x'), -1) \in \Omega \quad \text{for each } x' \text{ near } 0' \in \mathbb{R}^{n-1} \text{ if } t > 0 \text{ is small.} \tag{12.25}$$

This amounts to checking that if x' is near $0' \in \mathbb{R}^{n-1}$ and if $t \in (0, \infty)$ is small then we have $\varphi\big(x' - t\nabla\varphi(x')\big) < \varphi(x') + t$ which, in turn, follows by observing that (recall that φ is differentiable at points near $0'$)

$$\lim_{t\to 0^+} \frac{\varphi\big(x' - t\nabla\varphi(x')\big) - \varphi(x')}{t} = \frac{d}{dt}\big[\varphi\big(x' - t\nabla\varphi(x')\big)\big]\Big|_{t=0} = -|\nabla\varphi(x')|^2 < 1. \tag{12.26}$$

Thus (12.25) holds and, when considered together with the fact that $-h(x', \varphi(x'))$ is a unit vector which also points away from Ω (recall that this is the axis of the pseudo-ball with apex at $(x', \varphi(x'))$ which is contained in Ω^c) ultimately gives that

$$h(x', \varphi(x')) = \frac{(-\nabla\varphi(x'), 1)}{\sqrt{1 + |\nabla\varphi(x')|^2}} \quad \text{for each } x' \text{ near } 0' \in \mathbb{R}^{n-1}. \tag{12.27}$$

Note that since $\mathbb{R}^{n-1} \ni x' \mapsto (x', \varphi(x')) \in \partial\Omega$ is Lipschitz, and since $h \in \mathscr{C}^{\widetilde{\omega}}$ it follows that the mapping $x' \mapsto h(x', \varphi(x'))$ defined for x' near $0' \in \mathbb{R}^{n-1}$ belongs to $\mathscr{C}^{\widetilde{\omega}}$ as well. Moreover, (12.27) also shows that $h_n(x', \varphi(x')) \geq (1+M^2)^{-\frac{1}{2}}$, where $M > 0$ is the Lipschitz constant of φ, and

$$\partial_j \varphi(x') = -\frac{h_j(x', \varphi(x'))}{h_n(x', \varphi(x'))}, \qquad j = 1, ..., n-1, \tag{12.28}$$

granted that x' is near $0' \in \mathbb{R}^{n-1}$. Based on this it follows that $\nabla\varphi$ is of class $\mathscr{C}^{\widetilde{\omega}}$ near $0' \in \mathbb{R}^{n-1}$, where $\widetilde{\omega}$ is as in (12.2). Thus, φ is of class $\mathscr{C}^{1,\widetilde{\omega}}$ near $0' \in \mathbb{R}^{n-1}$. While this is a step in the right direction, more work is required in order to justify the stronger claim made in (12.21).

We wish to show that there exists $C > 0$ such that

$$|\nabla\varphi(x'_0) - \nabla\varphi(x'_1)| \leq C\omega(|x'_0 - x'_1|)$$

$$\text{whenever } x'_0 \text{ and } x'_1 \text{ are near } 0' \in \mathbb{R}^{n-1}. \tag{12.29}$$

Thanks to Lemma 10.6 we may, without loss of generality, assume that $(x'_0, \varphi(x'_0)) = (0', 0)$ and that $\nabla\varphi(x'_0) = 0'$. As such, matters are reduced to proving that

$$|\nabla\varphi(x'_1)| \leq C\,\omega(|x'_1|) \quad \text{ for } x'_1 \text{ near } 0'. \tag{12.30}$$

Since this is trivially true when $|\nabla\varphi(x'_1)| = 0$, it suffices to focus on the case when $|\nabla\varphi(x'_1)| \neq 0$. In this scenario, define

$$x'_2 := x'_1 + |x'_1| \frac{\nabla\varphi(x'_1)}{|\nabla\varphi(x'_1)|} \tag{12.31}$$

and note that, by the triangle inequality, $|x'_2| \leq 2|x'_1|$. As the point $(x'_2, \varphi(x'_2))$ lies on $\partial\Omega$, it does not belong to $\mathscr{G}^{\omega}_{a,b}((x'_1, \varphi(x'_1)), \pm h((x'_1, \varphi(x'_1))))$. As a consequence, we either have

$$\begin{aligned} &|(x'_2 - x'_1, \varphi(x'_2) - \varphi(x'_1))|\,\omega\big(|(x'_2 - x'_1, \varphi(x'_2) - \varphi(x'_1))|\big) \\ &\qquad\qquad \geq |h((x'_1, \varphi(x'_1))) \cdot (x'_2 - x'_1, \varphi(x'_2) - \varphi(x'_1))|, \end{aligned} \tag{12.32}$$

or

$$|h((x_1',\varphi(x_1')))\cdot(x_2'-x_1',\varphi(x_2')-\varphi(x_1'))|\geq b. \tag{12.33}$$

However, given that φ is Lipschitz, the latter eventuality never materializes if we choose x_1' sufficiently close to $0'$. Note that (12.31) forces $|(x_2'-x_1',\varphi(x_2')-\varphi(x_1'))|\leq\sqrt{1+M^2}|x_1'|$ where $M>0$ is the Lipschitz constant of φ. Since ω is increasing and satisfies the condition recorded in the last line of (8.11), we may write

$$\omega\big(|(x_2'-x_1',\varphi(x_2')-\varphi(x_1'))|\big)\leq C\,\omega\big(\sqrt{1+M^2}|x_1'|\big)\leq C\,\eta(\sqrt{1+M^2})\omega(|x_1'|), \tag{12.34}$$

for x_1' near $0'$ and x_2' as in (12.31). The bottom line of this portion of our analysis is that for some finite constant $C>0$

$$\begin{aligned}&\big|h((x_1',\varphi(x_1')))\cdot(x_2'-x_1',\varphi(x_2')-\varphi(x_1'))\big|\leq C\,\omega(|x_1'|)\\&\text{for } x_1' \text{ near } 0' \text{ and } x_2' \text{ as in (12.31).}\end{aligned} \tag{12.35}$$

In view of (12.27) and the fact that φ is Lipschitz, we obtain from (12.35) that

$$\begin{aligned}&\big|-\nabla\varphi(x_1')\cdot(x_2'-x_1')+\varphi(x_2')-\varphi(x_1')\big|\leq C|x_1'|\,\omega(|x_1'|)\\&\text{for } x_1' \text{ near } 0' \text{ and } x_2' \text{ as in (12.31).}\end{aligned} \tag{12.36}$$

This estimate further entails $|x_1'||\nabla\varphi(x_1')|\leq C|x_1'|\,\omega(|x_1'|)+C|\varphi(x_1')|+C|\varphi(x_2')|$ for some $C>0$ independent of x_1' near $0'$ (again, x_2' as in (12.31)). Let us now examine $|\varphi(x_1')|$. Given that the point $(x_1',\varphi(x_1'))$ lies on the boundary of Ω, it does not belong to $\mathscr{G}^{\omega}_{a,b}(0,\pm\mathbf{e}_n)$. Much as before, this necessarily implies

$|(x_1',\varphi(x_1'))|\,\omega\big(|(x_1',\varphi(x_1'))|\big)\geq|\varphi(x_1')|$. Since it is assumed that $\varphi(0')=0$ we further deduce that

$$\big|(x_1',\varphi(x_1'))\big|=(|x_1'|^2+(\varphi(x_1')-\varphi(0'))^2)^{\frac{1}{2}}\leq C|x_1'|, \tag{12.37}$$

by the Lipschitzianity of φ. Hence, ultimately, $|\varphi(x_1')|\leq C|x_1'|\,\omega(|x_1'|)$, by arguing as before. Likewise, $|\varphi(x_2')|\leq C|x_2'|\,\omega(|x_2'|)$ and since $|x_2'|\leq 2|x_1'|$, we see that $|\varphi(x_2')|\leq C|x_1'|\,\omega(|x_1'|)$. All in all, the above reasoning gives $|x_1'||\nabla\varphi(x_1')| \leq C|x_1'|\,\omega(|x_1'|) + C|\varphi(x_1')| + C|\varphi(x_2')| \leq C|x_1'|\,\omega(|x_1'|)$. Dividing the most extreme sides of this inequality by $|x_1'|$ then yields $|\nabla\varphi(x_1')| \leq C\,\omega(|x_1'|)$, as desired. This concludes the proof of (12.21) and, hence, Ω is of class $\mathscr{C}^{1,\omega}$ near x_*.

Consider now the scenario when the proper, open nonempty set $\Omega \subseteq \mathbb{R}^n$ is of class $\mathscr{C}^{1,\omega}$ near some boundary point $x_* \in \partial\Omega$, where ω is as in (8.11). In particular, $\omega : [0, R] \to [0, +\infty)$ is continuous, strictly increasing and such that $\omega(0) = 0$. The goal is to show that Ω satisfies a uniform hour-glass condition near x_* with shape function ω. To this end, based on Definition 10.1 and Lemma 10.6, there is no loss of generality in assuming that x_* is the origin in $\mathbb{R}^n$ and that if $(\mathcal{C}_{r,c}, \varphi)$ is the local chart near $0 \in \mathbb{R}^n$ then

$$\begin{array}{c}\text{the symmetry axis of the cylinder } \mathcal{C}_{r,c} \text{ is in the vertical direction } \mathbf{e}_n,\\ \varphi : \mathbb{R}^{n-1} \to \mathbb{R} \ \text{ is of class } \ \mathscr{C}^{1,\omega}, \quad \varphi(0') = 0, \quad \text{and} \quad \nabla\varphi(0') = 0'.\end{array} \tag{12.38}$$

Fix a constant $C \in (0, +\infty)$ with the property that

$$C \geq \sup_{\substack{x',y' \in \mathbb{R}^{n-1}\\ x' \neq y'}} \frac{|\nabla\varphi(x') - \nabla\varphi(y')|}{\omega(|x' - y'|)}. \tag{12.39}$$

The job at hand is to determine $b > 0$, depending only on r, c and φ, with the property that

$$\mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n) \subseteq \mathcal{C}_{r,c} \cap (\text{upper-graph of } \varphi), \tag{12.40}$$

$$\mathscr{G}^{\omega}_{a,b}(0, -\mathbf{e}_n) \subseteq \mathcal{C}_{r,c} \cap (\text{lower-graph of } \varphi). \tag{12.41}$$

Recall (8.5). Given that the mapping $t \mapsto t\,\omega(t)$ is increasing, it follows that $t_b \searrow 0$ as $b \searrow 0$. Consequently, we may select

$$b \in (0, R\,\omega(R)) \ \text{ small enough so that } \ t_{b/C} < \min\{r, c\}. \tag{12.42}$$

By part (i) in Lemma 8.1 such a choice ensures that $\mathscr{G}^{\omega}_{a,b}(0, \pm\mathbf{e}_n) \subseteq B(0, t_b) \subseteq \mathcal{C}_{r,c}$. Pick now an arbitrary point $x = (x', x_n) \in \mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n)$. Then, on the one hand, we have $C|x|\,\omega(|x|) < x_n < b$. On the other hand, (12.38) and the Mean Value Theorem ensure the existence of some $\theta = \theta(x') \in (0, 1)$

with the property that $\varphi(x') = x' \cdot (\nabla\varphi(\theta x') - \nabla\varphi(0'))$. This and the fact that φ is of class $\mathscr{C}^{1,\omega}$ then allow us to estimate $\varphi(x') \leq |x'||\nabla\varphi(\theta x') - \nabla\varphi(0')| \leq C|x'|\,\omega(|x'|) \leq C|x|\,\omega(|x|) < x_n$. This estimate shows that the point x belongs to the upper graph of the function φ. In summary, this discussion proves that (12.40) holds in the current setting. The same type of analysis as above (this time, writing $\varphi(x') \geq -C|x'|\,\omega(|x'|) \geq -C|x|\,\omega(|x|) > x_n$), shows that (12.41) also holds under these conditions. All in all, Ω satisfies a two-sided pseudo-ball condition at 0 with shape function ω, aperture C and height depending only on the $\mathscr{C}^{1,\omega}$ nature of Ω. This, of course, suffices to complete the proof of the theorem. $\square$

Chapter 13

A Historical Perspective on the Hopf-Oleinik Boundary Point Principle

The question of how the geometric properties of the boundary of a domain influence the behavior of a solution to a second-order elliptic equation is of fundamental importance and has attracted an enormous amount of attention. A significant topic, with distinguished pedigree, belonging to this line of research is the understanding of the sign of oblique directional derivatives of such a solution at boundary points. A celebrated result in this regard, known as the "Boundary Point Principle", states that an oblique directional derivative of a nonconstant $\mathscr{C}^2$ solution to a second-order, uniformly elliptic operator L in non-divergence form[1] with bounded coefficients, at an extremal point located on the boundary of the underlying domain $\Omega \subseteq \mathbb{R}^n$ is necessarily nonzero provided the domain is sufficiently regular at that point. Part of the importance of this result stems from its role in the development of the Strong Maximum Principle[2], as well as its applications to the issue of regularity near the boundary and uniqueness for a number of basic boundary value problems (such as Neumann, Robin, and mixed).

[1] As is well-known, the Boundary Point Principle fails in the class of divergence form second-order uniformly elliptic operators with bounded coefficients, even when these coefficients are continuous at the boundary point (cf. [29, p. 169], [82, p. 39], [30, Problem 3.9, pp. 49-50], [73]), though does hold if the coefficients are Hölder continuous at the boundary point – cf. [26].

[2] This is referred to in [82, p. 1] as a "bedrock result of the theory of second-order elliptic partial differential equations."

In the (by now) familiar version in which the regularity demand on the domain in question amounts to an interior ball condition, and when the second-order, non-divergence form, differential operator is uniformly elliptic and has bounded coefficients, this principle is due to E. Hopf and O.A. Oleinik who have done basic work on this topic in the early 1950's. However, the history of this problem is surprisingly rich, stretching back for more than a century and involving many contributors. Since the narrative of this endeavor does not appear to be well-known[3], below we attempt a brief survey of some of the main stages in the development of this topic.

Special cases of the Boundary Point Principle have been known for a long time since this contains, in particular, the fact that Green's function associated with a uniformly elliptic operator L in a domain Ω has a positive conormal derivative at boundary points provided $\partial\Omega$ and the coefficients of L are sufficiently regular. Some of the early references on this theme are the works of C. Neumann in [76] and A. Korn in [61] in the case of the Laplacian, and L. Lichtenstein [64] for more general operators.

In his pioneering 1910 paper [92][4], M.S. Zaremba has dealt with the case of the Laplacian in a three-dimensional domain Ω satisfying an interior ball condition at a point $x_0 \in \partial\Omega$ (cf. [92, Lemme, pp. 316-317]). His proof makes use of a barrier function, constructed with the help of Poisson's formula for harmonic functions in a ball. Concretely if, say, $B(0,r) \subseteq \Omega \subseteq \mathbb{R}^3$ and $x_0 \in \partial\Omega \cap \partial B(0,r)$, then Zaremba takes (cf. [92, p. 317])

$$v(x) := \frac{r^2 - |x|^2}{r} \int_{\partial B(0,r)} \frac{\psi(y)}{|x-y|^3}\, d\mathcal{H}^2(y), \qquad x \in \overline{B(0,r)}, \tag{13.1}$$

where ψ is a continuous, nonnegative function defined on $\partial B(0,r)$, which is zero near x_0 but otherwise

[3]For example, Zaremba's pioneering work at the beginning of the 20-th century is occasionally misrepresented as having been carried out in $\mathscr{C}^2$ domains when, in fact, in 1910 Zaremba has proved a Boundary Point Principle (for the Laplacian) in domains satisfying an interior ball condition at the point in question (a geometrical hypothesis which will remain the norm for the next 50 years).

[4]Zaremba's original motivation in this paper is the treatment of Dirichlet-Neumann mixed boundary value problems for the Laplacian. The nowadays familiar name "Zaremba's problem" has been eventually adopted in recognition of his early work in [92] (interestingly, in the preamble of this paper, Zaremba attributes the question of considering such a mixed boundary value problem to Wilhelm Wirtinger).

does not vanish identically. As such, the function in (13.1) is harmonic, nonnegative and vanishes at points on $\partial B(0,r)$ near x_0, and satisfies[5]

$$(-\frac{x_0}{r})\cdot(\nabla v)(x_0) = 2\int_{\partial B(0,r)} \frac{\psi(y)}{|x_0-y|^3}\,d\mathcal{H}^2(y) > 0. \tag{13.2}$$

These are the key features which virtually all subsequent generalizations based on barrier arguments will emulate in one form or another[6]. This being said, proofs based on other methods have been proposed over the years.

In 1932 G. Giraud managed to extend the Boundary Point Principle to a larger class of elliptic operators (containing the Laplacian), though this was done at the expense of imposing more restrictive conditions on the domain Ω. Specifically, in [31, Théorème 5, p. 343][7] he requires that Ω is of class $\mathscr{C}^{1,1}$ (cf. Definition 10.1) which, as indicated in the second part of Corollary 12.4, is equivalent to the requirement that Ω satisfies a uniform two-sided ball condition. The strategy adopted by Giraud in the proof of this result (cf. [31, pp. 343-346]) is essentially to reduce matters to the case of the Laplacian by freezing the coefficients and changing variables in a manner in which the Green function associated with the original differential operator may now be regarded as a perturbation of that for the Laplacian. Since the latter has an explicit formula, much as in the work by Zaremba, the desired conclusion follows. Shortly thereafter, in his 1933 paper [32], Giraud was able to sharpen the results he obtained earlier in [31] as to allow second-order elliptic operators whose top order coefficients are Hölder while the coefficients of the lower order terms are continuous[8], on domains of class $\mathscr{C}^{1,\alpha}$ where $\alpha \in (0,1)$; cf. [32, p. 50][9]. Giraud's proof of this more general result is a fairly

[5]In essence, this itself is a manifestation of the boundary point principle but in the very special case of a harmonic function in a ball.

[6]It is worth noting that Zaremba's approach works virtually verbatim for oblique derivative problems for the Laplacian.

[7]In the footnote on page 343 of his 1932 paper, Giraud's acknowledges on this occasion the earlier work done in 1931 by Marcel Brelot in his Thèse, pp. 27-28.

[8]The regularity conditions on the coefficients are not natural since, as is trivially verified, the class of differential operators for which the Boundary Point Principle holds is stable under multiplication by arbitrary (hence, possibly discontinuous) functions.

[9]Giraud's result is restated in [69, Theorem 3, IV, p. 7] for $\mathscr{C}^{1,\alpha}$ domains, though the proof given there is in the

laborious argument based on a change of variables (locally flattening the boundary).

Giraud's progress seems to have created a conundrum at this stage in the early development of the subject, namely there appeared to be two sets of conditions of geometric/analytic nature (which overlap but are otherwise unrelated) ensuring the validity of the Boundary Point Principle: on the one hand this holds for the Laplacian in domains satisfying an interior ball condition, while on the other hand this also holds for more general elliptic operators in domains of class $\mathscr{C}^{1,\alpha}$ with $\alpha \in (0,1)$[10].

A few years later, in 1937, motivated by the question of uniqueness for the Neumann problem for the Laplacian[11], M. Keldysch and M. Lavrentiev have proved in [56] a version of the Boundary Point Principle for the Laplacian in three-dimensional domains satisfying a more flexible property than the interior ball condition. Specifically, if $a, b \in (0,+\infty)$ and $\alpha \in (0,1]$, consider the three-dimensional, open, truncated paraboloid of revolution (about the z-axis) with apex at $0 \in \mathbb{R}^3$,

$$\mathscr{P}^{\alpha}_{a,b} := \big\{(x,y,z) \in \mathbb{R}^3 : \, a(x^2+y^2)^{\frac{1+\alpha}{2}} < z < b\big\}, \tag{13.3}$$

and say that $\Omega \subseteq \mathbb{R}$ satisfies an *interior paraboloid condition* at a boundary point $x_0 \in \partial\Omega$ provided one can place a congruent version of $\mathscr{P}^{\alpha}_{a,b}$ (for some choice of the exponent $\alpha \in (0,1]$ and the geometrical parameters $a, b > 0$) inside Ω in such a manner that the apex is repositioned at x_0. With this piece of terminology, M. Keldysch and M. Lavrentiev's 1937 result then states that the Boundary Point Principle holds for the Laplacian in any domain satisfying an interior paraboloid condition at the point in question. This extends Zaremba's 1910 work in [92] by allowing considerably more general domains and, at the same time, is more in line with the geometrical context in Giraud's

spirit of [43] and actually requires smoother boundaries.

[10]Typically, this is indicative of the fact that a more general phenomenon is at play. Alas, it will take about another 40 years for this issue to be resolved.

[11]This issue of uniqueness for the Neumann problem for the Laplacian has been raised by N. Gunther in his influential 1934 monograph on potential theory; cf. [35, Remarque, p. 99]. In this connection, we wish to note that in the 1967 English translation [36] of the original 1934 version of N. Gunther's book, this particular question has been omitted, and replaced by its solution given by M. Keldysch and M. Lavrentiev in [56].

1933 paper [32] since any domain of class $\mathscr{C}^{1,\alpha}$ with $\alpha \in (0,1)$ satisfies a paraboloid condition (for the same α in (13.3); e.g., this is implicit in the proof of Theorem 12.3). However, the conundrum described in the previous paragraph continued to persist.

As in Zaremba's approach, M. Keldysch and M. Lavrentiev's proof also relies upon the construction of a barrier function, albeit this is now adapted to the nature of the paraboloid (13.3). Specifically, in [56, p. 142] these authors consider following the barrier in $\mathscr{P}^{\alpha}_{a,b}$:

$$v(x,y,z) := z + \lambda\, r^{1+\beta} P_{1+\beta}(z/r), \qquad \forall\, (x,y,z) \in \mathscr{P}^{\alpha}_{a,b}, \tag{13.4}$$

where $\beta \in (0,\alpha)$, $\lambda > 0$ is a normalization constant, $r := \sqrt{x^2+y^2+z^2}$, and $P_{1+\beta}$ is the (regular, normalized) solution to Legendre's differential equation[12] of order $1+\beta$:

$$(1-t^2)\frac{d^2}{dt^2}P_{1+\beta}(t) - 2t\frac{d}{dt}P_{1+\beta}(t) + (1+\beta)(2+\beta)P_{1+\beta}(t) = 0. \tag{13.5}$$

Then β and λ may be chosen so that v in (13.5) has the same key features as in the earlier work of Zaremba. Of course, the case $\alpha = 1$ corresponds to Zaremba's interior ball condition.

As a corollary of their Boundary Point Principle, M. Keldysch and M. Lavrentiev then establish the uniqueness for the Neumann problem (classically formulated[13]) for a family of domains which contains all bounded domains of class $\mathscr{C}^{1,\alpha}$ with $\alpha \in (0,1)$. The issue whether this uniqueness result also holds for bounded domains of class $\mathscr{C}^1$ has subsequently become known as the Lavrentiev-Keldysch problem (cf. [60, p. 96]), and it will only be settled later. Momentarily fast-forwarding in time to 1981, it was N.S. Nadirashvili who in [72] proved a weaker version[14] of the Boundary Point Principle in bounded domains satisfying a global interior uniform cone condition (as discussed in

[12] A higher dimensional analogue of the Keldysch-Lavrentiev barrier requires considering Gegenbauer functions in place of solutions of (13.5).

[13] That is, the solution is assumed to be twice continuously differentiable inside the domain and continuous on the closure of the domain, with the normal derivative understood in as a one-sided directional derivative along the unit normal.

[14] Indeed, the Boundary Point Principle fails in the general class of Lipschitz domains; see [85, p. 4] for a simple counterexample in a two dimensional sector.

Definition 11.2) which nonetheless suffices to deduce uniqueness in the Neumann and oblique boundary value problems in such a setting[15] (cf. also [48, p. 307] for further refinements of Nadirashvili's theorem).

The coming of age of the work initiated by Zaremba in the 1910 is marked by the publication in 1952 of the papers [43], [78], in which E. Hopf[16] and O.A. Oleinik[17] have simultaneously and independently established a version of the Boundary Point Principle for domains satisfying an interior ball condition and for general, non-divergence form, uniformly elliptic operators with bounded coefficients[18]. In fact, Hopf and Oleinik's proofs differ only by their choice of barrier functions. In [43, p. 792], Hopf considered a barrier function in an annulus[19] given by

$$v(x) := e^{a|x|^2} - e^{ar^2}, \qquad \forall\, x \in B(0,r) \setminus \overline{B(0,r/2)}, \quad r > 0, \tag{13.6}$$

where $a > 0$ is a sufficiently large constant (chosen in terms of the coefficients of L)[20]. Oleinik took a different approach to the construction of a barrier and in [78, p. 696] considered the following function[21] defined in a ball:

$$v(x) := C_1 x_n + x_n^2 - C_2 \sum_{i=1}^{n-1} x_i^2 \qquad \forall\, x = (x_1, ..., x_n) \in B(r\mathbf{e}_n, r), \quad r > 0, \tag{13.7}$$

where $C_1, C_2 > 0$ are suitably chosen constants (depending on the size of the differential operator L).

[15]The crux of Nadirashvili's paper [72] is that, for domains satisfying a uniform cone condition, while the directional derivative of a supersolution of a uniformly elliptic differential operator in non-divergence form may vanish at an extremal point located on the boundary, it does not, however, vanish identically in any neighborhood of that point.

[16]The crucial observation Hopf makes in 1952 is that the comparison method he employed in his 1927 paper [42, Section I] may be used to establish, similarly yet independently of the Strong Maximum Principle itself, a remarkably versatile version of the Boundary Point Principle.

[17]Oleinik's paper was published two years before she defended her doctoral dissertation, entitled "Boundary-value problems for PDE's with small parameter in the highest derivative and the Cauchy problem in the large for non-linear equations" in 1954.

[18]Strictly speaking, both Hopf and Oleinik ask in [43], [78] that the coefficients of the differential operator in question are continuous, but their proofs go trough verbatim under the weaker assumption of boundedness.

[19]The idea of considering this type of region apparently originated with D. Gilbarg who used it in [28, pp. 312-313].

[20]An elegant alternative to Hopf's barrier function (13.6) in the same annulus is $\widetilde{v}(x) := |x|^{-\lambda} - r^{-\lambda}$ for a sufficiently large constant $\lambda > 0$; see the discussion in [63, § 1.3].

[21]Interestingly, in the limiting case $\alpha = \beta = 1$, the Keldysch-Lavrentiev barrier (13.4) becomes (given the known formula $P_2(t) = \frac{3}{2}t^2 - \frac{1}{2}$ for the second-order Lagrange polynomial) precisely $v(x, y, x) = z + z^2 - \frac{1}{2}(x^2 + y^2)$, which strongly resembles Oleinik's barrier (13.7) in the three-dimensional setting.

In this format, the Hopf-Oleinik Boundary Point Principle has become very popular and, even more than half a century later, is still routinely reproduced in basic text-books on partial differential equations (cf., e.g., [20], [82], [27], as well as the older monographs [30], [63], [69], [79]). However, the interior ball condition is unnecessarily restrictive and, as such, attempts were made to generalize Hopf and Oleinik's result (in a conciliatory manner with Giraud's 1933 result valid for domains of class $\mathscr{C}^{1,\alpha}$, $\alpha \in (0,1)$). Motivated by A.D. Aleksandrov's basic work in [1]-[6], in a series of papers beginning in the early 1970's (cf. [50], [49], [52], [53]) L.I. Kamynin and B.N. Khimchenko[22] succeeded[23] in extending the validity range of the Boundary Point Principle for general elliptic operators in non-divergence form with bounded coefficients to the class of domains satisfying an interior paraboloid condition, more general yet reminiscent of that considered by M. Keldysch and M. Lavrentiev in [56, p. 141]. More specifically, Kamynin and Khimchenko define in place of (13.3)

$$\mathscr{P}^{\omega}_{a,b} := \big\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \, a|x'|\,\omega(|x'|) < x_n < b\big\}\,, \tag{13.8}$$

where $a, b > 0$ and the (modulus of continuity, or) shape function $\omega \in \mathscr{C}^0\big([0,R]\big)$ is nonnegative, vanishes at the origin, and is required to satisfy certain differential/integral properties. For example, in [52], under the assumptions that

$$\omega \in \mathscr{C}^2\big((0,R)\big), \quad \omega'(t) \geq 0 \;\text{ and }\; \omega''(t) \leq 0 \;\text{ for every }\; t \in (0,R), \tag{13.9}$$

and granted that ω also satisfies a Dini integrability condition

$$\int_0^R \frac{\omega(t)}{t}\,dt < +\infty, \tag{13.10}$$

Kamynin and Khimchenko propose (cf. p. 84 in the English translation of [52]) the following exponential-type barrier which involves the above modulus of continuity

$$v(x) := x_n \exp\Big\{C_1 \int_0^{x_n} \frac{\widehat{\omega}(t)}{t}\,dt\Big\} - C_2\,|x|\,\omega(|x|), \quad \forall\, x = (x_1, ..., x_n) \in \mathscr{P}^{\omega}_{a,b}, \tag{13.11}$$

[22]Occasionally also spelled "Himčenko."
[23]Earlier, related results are due to R. Výborný in [90].

where $C_1, C_2 > 0$ are two suitably chosen constants. Here, $\widehat{\omega}$ is yet another modulus of continuity, satisfying the same type of conditions as in (13.9), and which is related to (in the terminology used in [52]) the nature of the degeneracy of the characteristic part of the differential operator L. A further refinement of this result, which applies to certain classes of differential operators with unbounded coefficients, has subsequently been worked out in [54] (cf. also [48]). Results of similar nature, but for domains satisfying an interior ball condition have been proved earlier by C. Pucci in [80], [81].

While the Dini condition (13.10) may not be omitted (cf. the discussion on pp. 85-88 in the English translation of [52]), the necessity of the differentiability conditions in (13.9) may be called into question. In this regard, see the discussion on p. 6 of [85], a paper in which M. Safonov proposes another approach to the Boundary Point Principle. His proof of [85, Theorem 1.8, p. 5] does not involve the use of a barrier function and, instead, is based on estimates for quotients u_2/u_1 of positive solutions of $Lu = 0$ in a Lipschitz domain Ω, which vanish on a portion of $\partial\Omega$. The main geometrical hypothesis in [85] is what the author terms interior Q-condition (replacing the earlier interior ball and paraboloid conditions), which essentially states that a region congruent to

$$Q := \big\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < R, \ \ 0 < x_n - |x'|\omega(|x'|) < R\big\} \tag{13.12}$$

may be placed inside Ω so as to make contact with the boundary at a desired point. In this scenario[24], Safonov retains (13.10) and, in place of (13.9), only assumes a monotonicity condition, to the effect that

$$\begin{aligned} &\omega : [0, R] \to [0, 1] \ \text{ is such that the mapping} \\ &[0, R] \ni t \mapsto t\omega(t) \in [0, R] \text{ is non-decreasing.} \end{aligned} \tag{13.13}$$

This being said, the method employed by Safonov requires that $u(x) = x_n$ is a solution of the operator

[24]Our notation is slightly different than that employed in [85], where the author works with $\psi(t) := t\omega(t)$ in place of ω.

L and, as such, he imposes the restriction that L is a differential operator without lower-order terms, i.e., $L = \sum\limits_{i,j=1}^{n} a^{ij}\partial_i\partial_j$, which is uniformly elliptic and has bounded coefficients. However, from the perspective of the Boundary Point Principle, a uniformly ellipticity condition is unnecessarily strong (as already noted in [52]) and, in fact, so is the boundedness assumption on the coefficients. Indeed, as is trivially verified, if the Boundary Point Principle is valid for a certain differential operator L, then it remains valid for the operator ψL where ψ is an arbitrary (thus, possibly unbounded) positive function.

The topic of Boundary Point Principles for partial differential equations remains an active area of research, with significant work completed in the recent past. See, for example, [85], [86], [73], [74], [75], [65], among others, and we have already commented on the contents of some of these papers. Here, we only wish to note that in [65, Theorem 4.1, p. 346] G.M. Libermann establishes a version of the Boundary Point Principle which, though weaker than that due to L.I. Kamynin and B.N. Khimchenko, has a conceptually simpler proof, which works in any $\mathscr{C}^1$ domain whose unit normal has a modulus of continuity satisfying a Dini integrability condition[25].

Finally, it should be mentioned that adaptations of this body of results to parabolic differential operators have been worked out by L. Nirenberg [77], L.I. Kamynin [47], L.I. Kamynin and B.N. Khimchenko [51], [55], to cite a few, and that a significant portion of the theory continues to hold for nonlinear partial differential equations (cf., e.g., [82] and the references therein).

[25]The class of domains considered in [65] is, however, not optimal.

Chapter 14

Boundary Point Principle for Semi-elliptic Operators with Singular Drift

Our main result in this chapter, formulated in Theorem 14.3 below, is a sharp version of the Hopf-Oleinik Boundary Point Principle. The proof presented here, which is a refinement of work recently completed in [12], is based on a barrier construction in a pseudo-ball (cf. (8.2)). This is done under less demanding assumptions on the shape function ω than those stipulated by Kamynin and Khimchenko in (13.9) and, at the same time, our pseudo-ball $\mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n)$ (cf. (8.4)) is a smaller set than the paraboloid $\mathscr{P}^{\omega}_{a,b}$ considered by Kamynin and Khimchenko in (13.8). Significantly, the coefficients of the differential operators for which our theorem holds are not necessarily bounded or measurable (in contrast to [43], [78], [49], [52], [53], and others), the matrix of top coefficients is only degenerately elliptic, and the coefficients of the lower-order terms are allowed to blow up at a rate related to the geometry of the domain[1]. Furthermore, by means of concrete counterexamples we show that that our result is sharp.

To set the stage, we first dispense of a number of preliminary matters.

Definition 14.1. *Let Ω be a nonempty, open, proper subset of $\mathbb{R}^n$, and fix a point $x_0 \in \partial\Omega$. We say*

[1] This addresses an issue raised in [86, p. 226].

that a vector $\vec{\ell} \in \mathbb{R}^n \setminus \{0\}$ `points inside` Ω `at` x_0 *provided there exists* $\varepsilon > 0$ *with the property that* $x_0 + t\vec{\ell} \in \Omega$ *whenever* $t \in (0, \varepsilon)$. *Given a function* $u \in \mathscr{C}^0(\Omega \cup \{x_0\})$ *and a vector* $\vec{\ell} \in \mathbb{R}^n \setminus \{0\}$ *pointing inside* Ω *at* x_0, *define the* `lower and upper directional derivatives of` u `at` x_0 `along` $\vec{\ell}$ *as*

$$\begin{aligned} D^{(inf)}_{\vec{\ell}} u(x_0) &:= \liminf_{t \to 0^+} \frac{u(x_0 + t\vec{\ell}) - u(x_0)}{t}, \quad \textit{and} \\ D^{(sup)}_{\vec{\ell}} u(x_0) &:= \limsup_{t \to 0^+} \frac{u(x_0 + t\vec{\ell}) - u(x_0)}{t}. \end{aligned} \tag{14.1}$$

Of course, in the same geometric setting as above, $D^{(\mathrm{inf})}_{\vec{\ell}} u(x_0)$, $D^{(\mathrm{sup})}_{\vec{\ell}} u(x_0)$ are meaningfully defined in $\overline{\mathbb{R}} := [-\infty, +\infty]$, there holds $D^{(\mathrm{inf})}_{\vec{\ell}} u(x_0) \leq D^{(\mathrm{sup})}_{\vec{\ell}} u(x_0)$ and, as a simple application of the Mean Value Theorem shows,

$$\left.\begin{array}{l} u \in \mathscr{C}^0(\Omega \cup \{x_0\}) \cap \mathscr{C}^1(\Omega) \text{ and the limit} \\ \nabla u(x_0) := \lim\limits_{t \to 0^+} (\nabla u)(x_0 + t\vec{\ell}) \ \text{ exists in } \mathbb{R}^n \end{array}\right\} \Rightarrow D^{(\mathrm{inf})}_{\vec{\ell}} u(x_0) = D^{(\mathrm{sup})}_{\vec{\ell}} u(x_0) = \vec{\ell} \cdot \nabla u(x_0). \tag{14.2}$$

Shortly, we shall need a suitable version of the Weak Minimum Principle. In order to facilitate the subsequent discussion, we first make a few definitions. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set and consider a second-order differential operator L in Ω:

$$L := -\sum_{i,j=1}^{n} a^{ij} \partial_i \partial_j + \sum_{i=1}^{n} b^i \partial_i, \qquad \text{where } \ a^{ij}, b^i : \Omega \to \mathbb{R}, \quad i, j \in \{1, ..., n\}. \tag{14.3}$$

Hence, L is in non-divergence form, without a zero-order term, and the reader is alerted to the presence of the minus sign in front of second-order part of L. In this context, recall that L is called *semi-elliptic* in Ω provided the coefficient matrix $A = (a^{ij})_{1 \leq i,j \leq n}$ is semi-positive definite at each point in Ω, i.e.,

$$\sum_{i,j=1}^{n} a^{ij}(x) \xi_i \xi_j \geq 0 \ \text{ for every } x \in \Omega \text{ and every } \xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n. \tag{14.4}$$

Clearly, the semi-ellipticity condition for L in Ω is equivalent to the requirement that, at each point in Ω, the symmetric part of the coefficient matrix $A := (a^{ij})_{1 \leq i,j \leq n}$, i.e., $\frac{1}{2}(A + A^\top)$ where $A^\top$ denotes the transpose of A, has only nonnegative eigenvalues. Also, we shall say that L (as above) is *non-degenerate along* $\xi^* = (\xi_1^*, ..., \xi_n^*) \in S^{n-1}$ in Ω provided

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i^*\xi_j^* > 0 \quad \text{for every } x \in \Omega. \tag{14.5}$$

For further use, let us also agree to call L *uniformly elliptic near* $x_0 \in \overline{\Omega}$ if there exists $r > 0$ such that

$$\inf_{x \in B(x_0,r)\cap\Omega} \inf_{\xi \in S^{n-1}} \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j > 0, \tag{14.6}$$

and simply *uniformly elliptic* provided

$$\inf_{x \in \Omega} \inf_{\xi \in S^{n-1}} \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j > 0. \tag{14.7}$$

Here is the variant of the Weak Minimum Principle alluded to above.

Proposition 14.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, nonempty, open set and assume that L is a second-order differential operator in non-divergence form (without a zero-order term) as in* (14.3) *which is semi-elliptic and non-degenerate along a vector $\xi^* = (\xi_1^*, ..., \xi_n^*) \in S^{n-1}$. In addition, suppose that the function*

$$\Omega \ni x \mapsto \frac{\sum_{i=1}^{n} b^i(x)\xi_i^*}{\sum_{i,j=1}^{n} a^{ij}(x)\xi_i^*\xi_j^*} \in \mathbb{R} \qquad \textit{is locally bounded from above in } \Omega. \tag{14.8}$$

Then for every real-valued function $u \in \mathscr{C}^2(\Omega)$ with the property that

$$(Lu)(x) \geq 0 \quad \textit{for every} \quad x \in \Omega, \tag{14.9}$$

it follows that

$$\inf_{x \in \Omega} u(x) = \inf_{x \in \partial\Omega} \Big(\liminf_{\Omega \ni y \to x} u(y) \Big). \tag{14.10}$$

In particular, if u is also continuous on $\overline{\Omega}$, then the minimum of u in $\overline{\Omega}$ is achieved on the topological boundary $\partial\Omega$, i.e.,

$$\min_{x \in \overline{\Omega}} u(x) = \inf_{x \in \Omega} u(x) = \min_{x \in \partial\Omega} u(x). \tag{14.11}$$

Proof. Though the proof of this result follows a well-established pattern, we include it for the sake of completeness. For starters, since $u \in \mathscr{C}^2(\Omega)$, by replacing a^{ij} with $\widetilde{a}^{ij} := \frac{1}{2}(a^{ij} + a^{ji})$, $1 \leq i, j \leq n$ (a transformation which preserves (14.4) and (14.8)), there is no loss of generality in assuming that the coefficient matrix $A = (a^{ij})_{1 \leq i,j \leq n}$ is symmetric at every point in Ω. Furthermore, observe that (14.10) is implied by the version of (14.11) in which Ω is replaced by any relatively compact subset of Ω, say, of the form $\Omega_k := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > 1/k\}$ where $k \in \mathbb{N}$, by passing to the limit $k \to +\infty$. Hence, there is no loss of generality in assuming that the function defined in (14.8) is actually globally bounded in Ω. With these adjustments in mind, the fact that

$$Lu > 0 \quad \text{in} \quad \Omega \Longrightarrow \min_{x \in \overline{\Omega}} u(x) = \min_{x \in \partial\Omega} u(x) \tag{14.12}$$

is then a simple consequence of the semi-positive definiteness of the (symmetric) matrix-coefficient (cf. (14.4)), and the Second Derivative Test for functions of class $\mathscr{C}^2$ (cf., e.g., [30, Theorem 3.1, p. 32]). Finally, in the case when the weaker condition (14.9) holds, one makes use of (14.12) with u replaced by $u + \varepsilon v$, where $\varepsilon > 0$ is arbitrary, the function $v : \Omega \to \mathbb{R}$ is given by (recall that $\xi^* \in S^{n-1}$ is as in (14.8))

$$v(x) := -e^{\lambda\, x \cdot \xi^*}, \qquad x \in \Omega, \tag{14.13}$$

and $\lambda \in (0, +\infty)$ is a fixed, sufficiently large constant. Concretely, since for every point $x \in \Omega$ we have

$$\begin{aligned} (Lv)(x) &= \lambda^2 \Big(\sum_{i,j=1}^{n} a^{ij}(x) \xi_i^* \xi_j^* \Big) e^{\lambda\, x \cdot \xi^*} - \lambda \Big(\sum_{i=1}^{n} b^i(x) \xi_i^* \Big) e^{\lambda\, x \cdot \xi^*} \\ &= \lambda\, e^{\lambda\, x \cdot \xi^*} \Big(\sum_{i,j=1}^{n} a^{ij}(x) \xi_i^* \xi_j^* \Big) \Big(\lambda - \frac{\sum_{i=1}^{n} b^i(x) \xi_i^*}{\sum_{i,j=1}^{n} a^{ij}(x) \xi_i^* \xi_j^*} \Big), \end{aligned} \tag{14.14}$$

it follows (cf. also (14.5)) that

$$\lambda > \sup_{x \in \Omega} \left(\frac{\sum_{i=1}^{n} b^i(x) \xi_i^*}{\sum_{i,j=1}^{n} a^{ij}(x) \xi_i^* \xi_j^*} \right) \Longrightarrow Lv > 0 \quad \text{in} \quad \Omega. \tag{14.15}$$

Hence, $\min\{(u+\varepsilon v)(x) : x \in \overline{\Omega}\} = \min\{(u+\varepsilon v)(x) : x \in \partial\Omega\}$ for each $\varepsilon > 0$, so (14.11) follows by letting $\varepsilon \to 0^+$. $\square$

Shortly, we shall also require the following simple algebraic lemma.

Lemma 14.2. *Let A be an $n \times n$ matrix, with real entries, which is semi-positive definite, i.e., it satisfies $(A\xi) \cdot \xi \geq 0$ for every $\xi \in \mathbb{R}^n$. Then, with $\mathrm{Tr}\,(A)$ denoting the trace of A, there holds*

$$\sup_{\xi \in S^{n-1}} [(A\xi) \cdot \xi] \leq \mathrm{Tr}\,(A). \tag{14.16}$$

Proof. Working with $\frac{1}{2}(A + A^\top)$ in place of A, there is no loss of generality in assuming that A is symmetric. Then there exists a unitary $n \times n$ matrix, U, and a diagonal $n \times n$ matrix, D, such that $A = U^{-1}DU$. If $\lambda_1, ..., \lambda_n$ are the entries on the diagonal of D, then $\lambda_i \geq 0$ for each $i \in \{1, ..., n\}$, and $\mathrm{Tr}\,(A) = \lambda_1 + \cdots + \lambda_n$. On the other hand, $\sup_{\xi \in S^{n-1}}[(A\xi) \cdot \xi] = \max\{\lambda_i : 1 \leq i \leq n\}$, so the desired conclusion follows. $\square$

As a final preliminary matter to discussing the theorem below, we make a couple of more definitions. Concretely, call a real-valued function f defined on an interval $I \subseteq \mathbb{R}$ *quasi-decreasing* provided there exists $C \in (0, +\infty)$ with the property that $f(t_1) \leq Cf(t_0)$ whenever $t_0, t_1 \in I$ are such that $t_0 \leq t_1$. Moreover, call f *quasi-increasing* if $-f$ is quasi-decreasing. Of course, the class of quasi-increasing (respectively, quasi-decreasing) functions contains the class of non-decreasing (respectively, non-increasing) functions, but the inclusion is strict[2]. In fact, if ϕ is non-decreasing and $C \geq 1$, then any function f with the property that $\phi \leq f \leq C\phi$ is quasi-increasing. Conversely, given a quasi-increasing function f, defining $\phi(t) := \inf_{s \geq t} f(s)$ yields a non-decreasing function for which $\phi \leq f \leq C\phi$ for some $C \geq 1$.

We are now prepared to state and prove the main result in this chapter.

[2]For example, if $\alpha > 0$ then $\omega(t) := (2 + \sin(t^{-1}))t^\alpha$, $t > 0$, is a quasi-increasing function which is not monotone in any interval of the form $(0, \varepsilon)$.

Theorem 14.3. *Suppose that Ω is a nonempty, proper, open subset of $\mathbb{R}^n$ and that $x_0 \in \partial\Omega$ is a point with the property that Ω satisfies an interior pseudo-ball condition at x_0. Specifically, assume that*

$$\mathscr{G}^{\omega}_{a,b}(x_0,h) = \{x \in B(x_0,R) :\, a|x-x_0|\,\omega(|x-x_0|) < h\cdot(x-x_0) < b\} \subseteq \Omega, \tag{14.17}$$

for some parameters $a,b,R \in (0,+\infty)$, direction vector $h=(h_1,...,h_n) \in S^{n-1}$, and a shape function $\omega : [0,R] \to [0,+\infty)$ exhibiting the following features:

$$\omega \text{ is continuous on } [0,R],\ \omega(t)>0 \text{ for } t \in (0,R], \quad \sup_{0<t\leq R}\Big(\frac{\omega(t/2)}{\omega(t)}\Big) < \infty, \tag{14.18}$$

$$\text{and the mapping } \ (0,R] \ni t \mapsto \frac{\omega(t)}{t} \in (0,+\infty) \ \text{ is quasi-decreasing.} \tag{14.19}$$

Also, consider a non-divergence form, second-order, differential operator (without a zero-order term)

$$L := -\sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j + \sum_{i=1}^{n} b^i\partial_i, \qquad a^{ij}, b^i : \Omega \longrightarrow \mathbb{R}, \quad 1 \leq i,j \leq n, \tag{14.20}$$

$$L \text{ semi-elliptic in } \Omega \text{ and non-degenerate along } h \in S^{n-1} \text{ in } \mathscr{G}^{\omega}_{a,b}(x_0,h). \tag{14.21}$$

In addition, suppose that there exists a real-valued function

$$\widetilde{\omega} \in \mathscr{C}^0\big([0,R]\big), \quad \widetilde{\omega}(t) > 0 \ \text{ for each } \ t \in (0,R], \ \text{ and } \ \int_0^R \frac{\widetilde{\omega}(t)}{t}\,dt < +\infty, \tag{14.22}$$

with the property that

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{\frac{\omega(|x-x_0|)}{|x-x_0|}\Big(\sum\limits_{i=1}^{n} a^{ii}(x)\Big)}{\frac{\widetilde{\omega}((x-x_0)\cdot h)}{(x-x_0)\cdot h}\Big(\sum\limits_{i,j=1}^{n} a^{ij}(x)h_ih_j\Big)} < \infty, \tag{14.23}$$

and

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{\max\Big\{0\,,\ \sum\limits_{i=1}^{n} b^i(x)h_i\Big\} + \Big(\sum\limits_{i=1}^{n}\max\big\{0\,,\,-b^i(x)\big\}\Big)\omega(|x-x_0|)}{\frac{\widetilde{\omega}((x-x_0)\cdot h)}{(x-x_0)\cdot h}\Big(\sum\limits_{i,j=1}^{n} a^{ij}(x)h_ih_j\Big)} < \infty. \tag{14.24}$$

Finally, suppose that $u : \Omega \cup \{x_0\} \to \mathbb{R}$ *is a function satisfying*

$$u \in \mathscr{C}^0(\Omega \cup \{x_0\}) \cap \mathscr{C}^2(\Omega), \tag{14.25}$$

$$(Lu)(x) \geq 0 \quad \textit{for each} \quad x \in \Omega, \tag{14.26}$$

$$u(x_0) < u(x) \quad \textit{for each} \quad x \in \Omega, \tag{14.27}$$

and fix a vector $\vec{\ell} \in S^{n-1}$ *satisfying the transversality condition*

$$\vec{\ell} \cdot h > 0. \tag{14.28}$$

Then $\vec{\ell}$ *points inside* Ω *at* x_0*, and there exist a compact subset* K *of* Ω *which depends only on the geometrical characteristics of* $\mathscr{G}^{\omega}_{a,b}(x_0, h)$*, and a constant* $\kappa > 0$ *which depends only on*

$$\begin{array}{l} \textit{the quantities in } (14.23)\text{-}(14.24), \quad \big(\inf_K u\big) - u(x_0), \\ \quad \vec{\ell} \cdot h, \quad \textit{and the pseudo-ball character of } \Omega \textit{ at } x_0, \end{array} \tag{14.29}$$

with the property that

$$(D^{(inf)}_{\vec{\ell}} u)(x_0) \geq \kappa. \tag{14.30}$$

Proof. We debut with a few comments pertaining to the nature of the functions ω, $\widetilde{\omega}$, and also make a suitable (isometric) change of variables in order to facilitate the subsequent discussion. First, the fact that $\widetilde{\omega}$ is continuous on $[0, R]$, positive on $(0, R]$ and satisfies Dini's integrability condition forces $\widetilde{\omega}(0) = 0$. Second, for further reference, let us fix a constant $\eta \in (0, +\infty)$ with the property that (cf. (14.19))

$$\frac{\omega(t_1)}{t_1} \leq \eta \frac{\omega(t_0)}{t_0} \qquad \text{whenever } 0 < t_0 \leq t_1 \leq R. \tag{14.31}$$

Third, from (14.23) and Lemma 14.2 it follows that there exists $C > 0$ with the property that

$$\omega(t) \leq C\, \widetilde{\omega}(t), \qquad \text{for all} \quad t \in [0, R]. \tag{14.32}$$

As a consequence of this and (14.22), we deduce that ω also satisfies Dini's integrability condition, i.e.,

$$\int_0^R \frac{\omega(t)}{t}\,dt < +\infty. \tag{14.33}$$

Moreover, it is also apparent from (14.18) and the Dini condition satisfied by ω that

$$\omega(0) = 0. \tag{14.34}$$

Fourth, we claim that there exist $M \in (0,+\infty)$ and $\gamma \in (1,+\infty)$ such that

$$(\eta\gamma)^{-1}\xi^{\gamma-1}\omega(\xi) \leq \int_0^\xi \omega(t)t^{\gamma-2}\,dt \leq M\xi^{\gamma-1}\omega(\xi), \qquad \forall\,\xi \in (0,R]. \tag{14.35}$$

To justify this claim, observe that if N stands for the supremum in the last condition in (14.18) then $N \in (0,+\infty)$ and

$$\omega(2^{-k}t) \leq N^k\omega(t), \qquad \forall\,t \in (0,R], \quad \forall\,k \in \mathbb{N}. \tag{14.36}$$

Next, fix a number $\gamma \in \mathbb{R}$ such that

$$\gamma > 1 + \max\{0\,,\,\log_2 N\}, \tag{14.37}$$

and recall that the function $(0,R] \ni t \mapsto \omega(t)/t \in (0,+\infty)$ is quasi-increasing. Then, if $\eta \in (0,+\infty)$ is as in (14.31), using the fact that $\gamma > 1$ as well as the estimates in (14.36)-(14.37), for every $\xi \in (0,R]$ we may write

$$\begin{aligned}
\int_0^\xi \omega(t)t^{\gamma-2}\,dt &= \sum_{k=0}^{+\infty}\int_{2^{-k-1}\xi}^{2^{-k}\xi}\frac{\omega(t)}{t}t^{\gamma-1}\,dt \leq \sum_{k=0}^{+\infty}(2^{-k}\xi)^{\gamma-1}\int_{2^{-k-1}\xi}^{2^{-k}\xi}\frac{\omega(t)}{t}\,dt \\
&\leq \eta\sum_{k=0}^{+\infty}(2^{-k}\xi)^{\gamma-1}\frac{\omega(2^{-k-1}\xi)}{2^{-k-1}\xi}2^{-k-1}\xi = \eta\xi^{\gamma-1}\sum_{k=0}^{+\infty}2^{-k(\gamma-1)}\omega(2^{-k-1}\xi) \\
&\leq \eta\xi^{\gamma-1}\sum_{k=0}^{+\infty}2^{-k(\gamma-1)}N^{k+1}\omega(\xi) = N\eta\,\xi^{\gamma-1}\omega(\xi)\Big(\sum_{k=0}^{+\infty}2^{-k(\gamma-1)}2^{k\log_2 N}\Big) \\
&= \eta N\Big(\sum_{k=0}^{+\infty}2^{-k(\gamma-1-\log_2 N)}\Big)\xi^{\gamma-1}\omega(\xi) = \frac{\eta N}{1-2^{-\gamma+1+\log_2 N}}\xi^{\gamma-1}\omega(\xi).
\end{aligned} \tag{14.38}$$

Thus, the upper-bound for the integral in (14.35) is proved with

$$M := \frac{\eta N}{1 - 2^{-\gamma+1+\log_2 N}} \in (0, +\infty). \tag{14.39}$$

Since the lower bound is a direct consequence of (14.31), this completes the proof of (14.35).

Continuing our series of preliminary matters, let U be an $n \times n$ unitary matrix (with real entries) with the property that $Uh = \mathbf{e}_n$ and define an isometry of $\mathbb{R}^n$ by setting $\mathcal{R}x := U(x - x_0)$ for every $x \in \mathbb{R}^n$. Introduce $\widetilde{\Omega} := \mathcal{R}(\Omega)$. Then if

$$\big(\widetilde{a}^{ij}(y)\big)_{1 \le i,j \le n} := U\big[\big(a^{ij}(\mathcal{R}^{-1}y)\big)_{1 \le i,j \le n}\big]U^{-1}, \qquad \forall\, y \in \widetilde{\Omega}, \tag{14.40}$$

$$\big(\widetilde{b}^{i}(y)\big)_{1 \le i \le n} := U\big[\big(b^{i}(\mathcal{R}^{-1}y)\big)_{1 \le i \le n}\big], \qquad \forall\, y \in \widetilde{\Omega}, \tag{14.41}$$

and if we consider the differential operator in $\widetilde{\Omega}$ given by

$$\widetilde{L} := -\sum_{i,j=1}^{n} \widetilde{a}^{ij}(y)\partial_{y_i}\partial_{y_j} + \sum_{i=1}^{n} \widetilde{b}^{i}\partial_i, \tag{14.42}$$

then $\widetilde{L}$ satisfies properties analogous to L (relative to the new geometrical context), and

$$\widetilde{L}(u \circ \mathcal{R}^{-1}) = (Lu) \circ \mathcal{R}^{-1}. \tag{14.43}$$

Furthermore, $\mathcal{R}(\mathscr{G}^{\omega}_{a,b}(x_0, h)) = \mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n)$ by (8.8). To summarize, given that both the hypotheses and the conclusion in the statement of the theorem transform covariantly under the change of variables $y = \mathcal{R}x$, there is no loss of generality in assuming that, to begin with, x_0 is the origin in $\mathbb{R}^n$ and that $h = \mathbf{e}_n \in S^{n-1}$. In this setting, the transversality condition (14.28) becomes

$$\vec{\ell} \cdot \mathbf{e}_n > 0, \tag{14.44}$$

while the semi-ellipticity condition on L and non-degeneracy condition on L along $h \in S^{n-1}$ read

$$\inf_{x \in \mathscr{G}^{\omega}_{a,b}(0,\mathbf{e}_n)} \inf_{\xi \in S^{n-1}} \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \ge 0 \quad \text{and} \quad a^{nn}(x) > 0 \quad \forall\, x \in \mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n). \tag{14.45}$$

Going further, for each real number r set

$$[r]_{\oplus} := \max\{r, 0\} \quad \text{ and } \quad [r]_{\ominus} := \max\{-r, 0\}. \tag{14.46}$$

Then, as far as how (14.23)-(14.24) transform under the indicated change of variables, we note that after possibly decreasing the value of R, matters may be arranged so that

$$\sum_{i=1}^{n} a^{ii}(x) \leq \Lambda_0 \frac{|x|\widetilde{\omega}(x_n)}{x_n \omega(|x|)} a^{nn}(x), \qquad \forall\, x \in \mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n), \tag{14.47}$$

$$\sum_{i=1}^{n} [b^i(x)]_{\ominus} \leq \Lambda_1 \frac{\widetilde{\omega}(x_n)}{x_n \omega(|x|)} a^{nn}(x), \qquad \forall\, x \in \mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n), \tag{14.48}$$

$$[b^n(x)]_{\oplus} \leq \Lambda_2 \frac{\widetilde{\omega}(x_n)}{x_n} a^{nn}(x), \qquad \forall\, x \in \mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n), \tag{14.49}$$

for some constants $\Lambda_0, \Lambda_1, \Lambda_2 \in (0, +\infty)$.

We are now ready to begin the proof in earnest. For starters, we note that by eventually increasing the value of $a > 0$ and decreasing the value of $b > 0$ we may assume that

$$\overline{\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)} \setminus \{0\} \subseteq \Omega, \qquad \forall\, b_* \in (0, b]. \tag{14.50}$$

To proceed, fix $b_* \in (0, b]$ and, with $\gamma \in (1, +\infty)$ as in (14.37) and for two finite constants $C_0, C_1 > 0$ to be specified later, consider the barrier function

$$v(x) := x_n + C_0 \int_0^{x_n} \int_0^{\xi} \frac{\widetilde{\omega}(t)}{t}\, dt\, d\xi - C_1 \int_0^{|x|} \int_0^{\xi} \frac{\omega(t)}{t} \Big(\frac{t}{\xi}\Big)^{\gamma-1} dt\, d\xi, \tag{14.51}$$

for every $x = (x_1, ..., x_n) \in \overline{\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)}$. Since ω, $\widetilde{\omega}$ are continuous and satisfy Dini's integrability condition, it follows that v is well-defined and, in fact,

$$v \in \mathscr{C}^2\big(\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)\big) \cap \mathscr{C}^0\big(\overline{\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)}\big). \tag{14.52}$$

Moreover, a direct computation gives that for each $x \in \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)$ we have

$$\partial_j v(x) = \delta_{jn} + C_0 \delta_{jn} \int_0^{x_n} \frac{\widetilde{\omega}(t)}{t}\, dt - C_1 \frac{x_j}{|x|} \int_0^{|x|} \frac{\omega(t)}{t} \Big(\frac{t}{|x|}\Big)^{\gamma-1} dt, \quad 1 \leq j \leq n, \tag{14.53}$$

and, further, for each $i, j \in \{1, ..., n\}$,

$$\partial_i \partial_j v(x) = C_0 \delta_{in} \delta_{jn} \frac{\widetilde{\omega}(x_n)}{x_n} - C_1 \Big[\tfrac{\delta_{ij}}{|x|^\gamma} - \gamma \tfrac{x_i x_j}{|x|^{\gamma+2}} \Big] \int_0^{|x|} \omega(t) t^{\gamma-2} dt - C_1 \frac{x_i x_j}{|x|^2} \frac{\omega(|x|)}{|x|} \tag{14.54}$$

where δ_{ij} is the usual Kronecker symbol. Hence, by combining (14.20) with (14.53)-(14.54), we arrive at the conclusion that

$$(Lv)(x) = I + II + III, \qquad \forall\, x \in \mathscr{G}^\omega_{a,b_*}(0, \mathbf{e}_n), \tag{14.55}$$

where, for each $x = (x_1, ..., x_n) \in \mathscr{G}^\omega_{a,b_*}(0, \mathbf{e}_n)$ we have set

$$I \;:=\; I' + I'' \;\text{ with }\; I' := C_1 \Big(\sum_{i=1}^n a^{ii}(x) \Big) |x|^{-\gamma} \int_0^{|x|} \omega(t) t^{\gamma-2}\, dt \;\text{ and} \tag{14.56}$$

$$I'' := C_1 \Big(\sum_{i,j=1}^n a^{ij}(x) \tfrac{x_i}{|x|} \tfrac{x_j}{|x|} \Big) \Big(\frac{\omega(|x|)}{|x|} - \gamma |x|^{-\gamma} \int_0^{|x|} \omega(t) t^{\gamma-2}\, dt \Big), \tag{14.57}$$

$$II \;:=\; -C_0\, a^{nn}(x) \frac{\widetilde{\omega}(x_n)}{x_n}, \tag{14.58}$$

$$III \;:=\; b^n(x) + C_0 b^n(x) \int_0^{x_n} \frac{\widetilde{\omega}(t)}{t}\, dt - C_1 \Big(\sum_{i=1}^n b^i(x) \tfrac{x_i}{|x|} \Big) \int_0^{|x|} \frac{\omega(t)}{t} \Big(\frac{t}{|x|} \Big)^{\gamma-1} dt. \tag{14.59}$$

As a preamble to estimating I, II, III above, we make a couple of preliminary observations. First note that since $C_1 \geq 0$, ω is nonnegative, and L is semi-elliptic, we have

$$I'' \leq C_1 \Big(\sum_{i,j=1}^n a^{ij}(x) \tfrac{x_i}{|x|} \tfrac{x_j}{|x|} \Big) \frac{\omega(|x|)}{|x|} \leq C_1 \Big(\sum_{i=1}^n a^{ii}(x) \Big) \frac{\omega(|x|)}{|x|}, \quad \forall\, x \in \mathscr{G}^\omega_{a,b_*}(0, \mathbf{e}_n). \tag{14.60}$$

where the last inequality above is based on Lemma 14.2. Second, for every point $x \in \mathscr{G}^\omega_{a,b_*}(0, \mathbf{e}_n)$, estimate (14.35) used with $\xi := |x| \in (0, R)$ gives that

$$|x|^{-\gamma} \int_0^{|x|} \omega(t) t^{\gamma-2}\, dt \leq M \frac{\omega(|x|)}{|x|}, \tag{14.61}$$

where the constant $M \in (0, +\infty)$ is as in (14.39). Consequently,

$$I' \leq M C_1 \Big(\sum_{i=1}^n a^{ii}(x) \Big) \frac{\omega(|x|)}{|x|}, \qquad \forall\, x \in \mathscr{G}^\omega_{a,b_*}(0, \mathbf{e}_n). \tag{14.62}$$

In concert with the above observations, formulas (14.56)-(14.59) then allow us to conclude that (recall notation introduced in (14.46)) for every $x \in \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)$,

$$I \leq C_1(1+M)\Big(\sum_{i=1}^{n} a^{ii}(x)\Big)\frac{\omega(|x|)}{|x|}, \qquad II \leq -C_0 a^{nn}(x)\frac{\widetilde{\omega}(x_n)}{x_n}, \tag{14.63}$$

$$III \leq [b^n(x)]_{\oplus}\Big(1 + C_0\int_0^{x_n}\frac{\widetilde{\omega}(t)}{t}\,dt\Big) + C_1 M\Big(\sum_{i=1}^{n}[b^i(x)]_{\ominus}\Big)\omega(|x|), \tag{14.64}$$

where we have also used (14.61) when deriving the last estimate above. Thus, on account of (14.55), (14.62), (14.63), and (14.31), for every $x \in \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)$ we may estimate

$$\begin{aligned}(Lv)(x) \;\leq\; & \frac{\widetilde{\omega}(x_n)}{x_n}a^{nn}(x)\Big\{C_1(1+M)\Big(\frac{\sum_{i=1}^{n} a^{ii}(x)}{a^{nn}(x)}\Big)\frac{x_n\omega(|x|)}{|x|\widetilde{\omega}(x_n)} - C_0\Big\} \\ & + \frac{\widetilde{\omega}(x_n)}{x_n}a^{nn}(x)\Big\{\frac{x_n[b^n(x)]_{\oplus}}{\widetilde{\omega}(x_n)a^{nn}(x)}\Big(1 + C_0\int_0^{x_n}\frac{\widetilde{\omega}(t)}{t}\,dt\Big) \\ & \qquad + C_1 M\frac{x_n\omega(|x|)\Big(\sum_{i=1}^{n}[b^i(x)]_{\ominus}\Big)}{\widetilde{\omega}(x_n)a^{nn}(x)}\Big\}.\end{aligned} \tag{14.65}$$

In turn, (14.65) and (14.47)-(14.49) permit us to further estimate, for each $x \in \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)$,

$$\begin{aligned}(Lv)(x) \;\leq\; & \frac{\widetilde{\omega}(x_n)}{x_n}a^{nn}(x)\Big\{C_1(1+M)\Lambda_0 - C_0\Big\} \\ & + \frac{\widetilde{\omega}(x_n)}{x_n}a^{nn}(x)\Big\{\Lambda_2\Big(1 + C_0\int_0^{x_n}\frac{\widetilde{\omega}(t)}{t}\,dt\Big) + C_1 M\Lambda_1\Big\} \\ \leq\; & \frac{\widetilde{\omega}(x_n)}{x_n}a^{nn}(x)\Big\{C_1(\Lambda_0 + M\Lambda_0 + M\Lambda_1) + \Lambda_2 - C_0\Big(1 - \Lambda_2\int_0^{b_*}\frac{\widetilde{\omega}(t)}{t}\,dt\Big)\Big\}.\end{aligned} \tag{14.66}$$

We shall return to (14.66) momentarily. For the time being, we wish to estimate the barrier function on the round portion of the boundary of the pseudo-ball. To this end, let us note from (8.2) that if $x = (x_1, ..., x_n) \in \partial\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n) \setminus \{x \in \mathbb{R}^n : x_n = b_*\}$ then, given that ω is continuous, we have $x_n = a\omega(|x|)|x|$ which further implies

$$\begin{aligned}x_n + C_0\int_0^{x_n}\int_0^{\xi}\frac{\widetilde{\omega}(t)}{t}\,dt\,d\xi \;&\leq\; x_n\Big(1 + C_0\int_0^{x_n}\frac{\widetilde{\omega}(t)}{t}\,dt\Big) \\ &=\; a\omega(|x|)|x|\Big(1 + C_0\int_0^{x_n}\frac{\widetilde{\omega}(t)}{t}\,dt\Big).\end{aligned} \tag{14.67}$$

Moreover, since $\omega(t)/t \geq \eta^{-1}\omega(|x|)/|x|$ for every $t \in (0,|x|)$ (cf. (14.31)), we may also write

$$\int_0^{|x|}\int_0^{\xi} \frac{\omega(t)}{t}\Big(\frac{t}{\xi}\Big)^{\gamma-1} dt\, d\xi \geq \eta^{-1}\frac{\omega(|x|)}{|x|}\int_0^{|x|}\int_0^{\xi}\Big(\frac{t}{\xi}\Big)^{\gamma-1} dt\, d\xi = \frac{|x|\omega(|x|)}{2\eta\gamma}. \tag{14.68}$$

Together, (14.51) and (14.67)-(14.68) give that for each

$x \in \partial\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n) \setminus \{x = (x_1,...,x_n) \in \mathbb{R}^n : x_n = b_*\}$ we have

$$v(x) \leq \Big(a - \frac{C_1}{2\eta\gamma} + aC_0\int_0^{b_*}\frac{\widetilde{\omega}(t)}{t}\, dt\Big)|x|\omega(|x|). \tag{14.69}$$

At this stage, we are ready to specify the constants $C_0, C_1 \in (0,+\infty)$ appearing in (14.51), in a manner consistent with the format of (14.66), (14.69) and which suits the goals we have in mind. Turning to details, we start by fixing

$$C_1 > 2a\eta\gamma \quad \text{and} \quad C_0 > 2\big[C_1(\Lambda_0 + M\Lambda_0 + M\Lambda_1) + \Lambda_2\big], \tag{14.70}$$

then, using the Dini integrability condition satisfied by $\widetilde{\omega}$, select $b_* \in (0,b]$ sufficiently small so that

$$\int_0^{b_*}\frac{\widetilde{\omega}(t)}{t}\, dt < \frac{1}{2\Lambda_2} \quad \text{and} \quad \int_0^{b_*}\frac{\widetilde{\omega}(t)}{t}\, dt < \frac{C_1 - 2a\eta\gamma}{2a\eta\gamma C_0}. \tag{14.71}$$

Then (14.66) together with the second condition in (14.70) and the first condition in (14.71) ensure that

$$Lv \leq 0 \quad \text{in} \quad \mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n). \tag{14.72}$$

Furthermore, the second condition in (14.71) is designed (cf. (14.69)) so that we also have

$$v \leq 0 \quad \text{on} \quad \partial\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n) \setminus \{x = (x_1,...,x_n) \in \mathbb{R}^n : x_n = b_*\}. \tag{14.73}$$

Having specified the constants C_0 and C_1 (in the fashion described above) finishes the process of defining the barrier function v, initiated in (14.51). With this task concluded, we proceed by considering the compact subset of Ω given by

$$K := \{x = (x_1,...,x_n) \in \overline{\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)} : x_n = b_*\}, \tag{14.74}$$

and note that (14.27) (and since u is continuous, hence attains its infimum on compact subsets of Ω) entails

$$u(x_0) < \inf_K u. \tag{14.75}$$

Thanks to (14.27), (14.73) and (14.75), we may then choose $\varepsilon > 0$ for which

$$\varepsilon\big(\sup_K |v|\big) < \big(\inf_K u\big) - u(x_0), \tag{14.76}$$

(hence ε depends only on the quantities listed in (14.29)) so that, on the one hand,

$$0 \leq u(x) - u(x_0) - \varepsilon v(x) \quad \text{for every } x \in \partial\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n). \tag{14.77}$$

On the other hand, from (14.72) and (14.26) we obtain (recall that L annihilates constants)

$$L(u - u(x_0) - \varepsilon v) \geq 0 \quad \text{in } \; \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n). \tag{14.78}$$

With the estimates (14.77)-(14.78) in hand, and keeping in mind (14.52) plus the fact that the function u belongs to $\mathscr{C}^0(\overline{\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)}) \cap \mathscr{C}^2(\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n))$, bring in the Weak Minimum Principle presented in Proposition 14.1. This is used in the nonempty, bounded, open subset $\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)$ of $\mathbb{R}^n$ and with the vector $\mathbf{e}_n$ playing the role of $\xi^* \in S^{n-1}$ from (14.8). Indeed, granted (14.45), it follows that L is non-degenerate along $\mathbf{e}_n \in S^{n-1}$ in $\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)$ and, thanks to (14.49), the analogue of condition (14.8) is valid in the current setting. The bottom line is that Proposition 14.1 applies, and gives

$$u - u(x_0) - \varepsilon v \geq 0 \quad \text{in } \overline{\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)}. \tag{14.79}$$

Given that both $u - u(x_0)$ and v vanish at the point $x_0 = 0 \in \partial\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)$, this shows that

$$u - u(x_0) - \varepsilon v \in \mathscr{C}^0(\overline{\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)}) \text{ has a global minimum at } x_0 = 0. \tag{14.80}$$

On the other hand, condition (14.44) and the fact that ω continuously vanishes at the origin (see (14.34)) imply the existence of some $t_* \in (0, b_*)$ with the property that $\omega(t) < \vec{\ell} \cdot \mathbf{e}_n / a$ for every $t \in (0, t_*)$. In turn, such a choice of t_* ensures that (cf. (8.2))

$$t\,\vec{\ell} \in \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n) \quad \text{for every} \;\; t \in (0, t_*). \tag{14.81}$$

In particular, $\vec{\ell}$ points in Ω at x_0 (cf. Definition 14.1), and from (14.1), (14.80)-(14.81) we obtain

$$D^{(\mathrm{inf})}_{\vec{\ell}}(u - u(x_0) - \varepsilon v)(x_0) \geq 0. \tag{14.82}$$

Now (14.53) gives

$$\nabla v(x_0) := \lim_{\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n) \ni x \to 0} (\nabla v)(x) = \mathbf{e}_n, \tag{14.83}$$

hence

$$(D^{(\mathrm{inf})}_{\vec{\ell}} v)(x_0) = (D^{(\mathrm{sup})}_{\vec{\ell}} v)(x_0) = \vec{\ell} \cdot \nabla v(x_0) = \vec{\ell} \cdot \mathbf{e}_n, \tag{14.84}$$

by (14.52) and the discussion in (14.2). In turn, (14.82)-(14.83) and (14.84) further allow us to conclude that

$$(D^{(\mathrm{inf})}_{\vec{\ell}} u)(x_0) \geq \varepsilon \vec{\ell} \cdot \nabla v(x_0) = \varepsilon\, \vec{\ell} \cdot \mathbf{e}_n > 0, \tag{14.85}$$

where the last inequality is a consequence of (14.44). Choosing $\kappa := \varepsilon\, \vec{\ell} \cdot \mathbf{e}_n > 0$ then yields (14.30), finishing the proof of the theorem. □

We continue with a series of comments relative to Theorem 14.3 and its proof.

Remark 14.1.

(i) As we will discuss in detail later, Theorem 14.3 is sharp. A slightly more versatile result is obtained by replacing Ω by $U \cap \Omega$ in (14.25)-(14.27)*, where $U \subseteq \mathbb{R}^n$ is some open neighborhood*

of $x_0 \in \partial\Omega$. *Of course, Theorem 14.3 itself implies such an improvement simply by invoking it with* Ω *substituted by* $U \cap \Omega$ *throughout.*

(ii) Trivially, the last condition in (14.18) *is satisfied if the function* $\omega : [0, R] \to [0, +\infty)$ *has the property that*

$$\exists\, m \in \mathbb{R} \text{ such that } (0, R] \ni t \mapsto t^m \omega(t) \in (0, +\infty) \text{ is quasi-increasing,} \tag{14.86}$$

hence, in particular, if ω *itself is quasi-increasing. Corresponding to the class of function introduced in* (1.18)*, the shape function* $\omega_{\alpha,\beta}$ *satisfies all properties displayed in* (14.18)-(14.19) *for all* $\alpha \in (0, 1]$ *and* $\beta \in \mathbb{R}$. *However,* $\omega_{0,\beta}$ *fails to satisfy the Dini integrability condition for* $\beta \geq -1$ *(while still meeting the other conditions).*

(iii) If $\omega : (0, R] \to (0, +\infty)$ *is such that the map* $(0, R] \ni t \mapsto \omega(t)/t \in (0, +\infty)$ *is quasi-decreasing and* $\sup\limits_{0<t\leq R}\Big(\frac{\omega(t/2)}{\omega(t)}\Big) < +\infty$, *then for every* $c \in (1, +\infty)$ *we also have* $\sup\limits_{0<t\leq R}\Big(\frac{\omega(t/c)}{\omega(t)}\Big) < +\infty$. *Based on this observation, one may then verify without difficulty that if* $\omega : [0, R] \to [0, +\infty)$ *satisfies the conditions in* (14.18)-(14.19) *then, for each fixed* $\theta \in (0, 1)$, *so does the function* $[0, R] \ni t \mapsto \omega(t^\theta) \in [0, +\infty)$. *Furthermore, this function satisfies Dini's integrability condition if* ω *does.*

(iv) The amplitude parameter $a > 0$ *used in defining the pseudo-ball* $\mathscr{G}^{\omega}_{a,b}(x_0, h)$ *plays only a minor role since this may, in principle, be absorbed as a multiplicative factor into the shape function* ω *(thus, reducing matters to the case when*

$a = 1$*). Nonetheless, working with a generic amplitude adds a desirable degree of flexibility in the proof of Theorem 14.3.*

(v) It is instructive to note that if $\omega : [0, R] \to [0, +\infty)$ *satisfies* (14.19) *as well as the first*

two properties listed in (14.18)*, and is such that* (14.35) *holds for some* $M \in (0, +\infty)$ *and* $\gamma \in (0, +\infty)$*, then actually the last condition in* (14.18) *is also valid. Indeed, using* (14.31)*, for each* $\xi \in (0, R]$ *we may estimate*

$$\begin{aligned} M\xi^{\gamma-1}\omega(\xi) &\geq \int_0^{\xi} \frac{\omega(t)}{t} t^{\gamma-1}\, dt \geq \int_0^{\xi/2} \frac{\omega(t)}{t} t^{\gamma-1}\, dt \\ &\geq \eta^{-1} \frac{\omega(\xi/2)}{\xi/2} \int_0^{\xi/2} t^{\gamma-1}\, dt = \frac{1}{\gamma\eta} \frac{\omega(\xi/2)}{\xi/2} \Big(\frac{\xi}{2}\Big)^{\gamma} \\ &= \frac{1}{\gamma 2^{\gamma-1}\eta} \xi^{\gamma-1}\omega(\xi/2), \end{aligned} \tag{14.87}$$

which entails

$$\sup_{0<\xi\leq R} \Big(\frac{\omega(\xi/2)}{\omega(\xi)}\Big) \leq \gamma 2^{\gamma-1} M\eta < +\infty. \tag{14.88}$$

We next prove a technical result (refining earlier work in [10]), which is going to be useful in the proof of Theorem 14.5 below.

Proposition 14.4. *Let* $R \in (0, +\infty)$ *and assume that* $\omega : [0, R] \to [0, +\infty)$ *is a continuous function with the property that* $\omega(t) > 0$ *for each* $t \in (0, R]$*. In addition, assume that* ω *satisfies a Dini condition and is quasi-increasing, i.e.,*

$$\int_0^R \frac{\omega(t)}{t}\, dt < +\infty \quad \text{and} \quad \omega(t_1) \leq \eta\, \omega(t_2) \ \text{ whenever } t_1, t_2 \in [0, R] \text{ are such that } t_1 \leq t_2, \tag{14.89}$$

for some fixed constant $\eta \in (0, +\infty)$*. Consider*

$$M := \max\big\{\omega(t) : t \in [0, R]\big\}, \qquad t_o := \min\big\{t \in [0, R] : \omega(t) = M\big\}, \tag{14.90}$$

and denote by $\theta_* \in (0, 1)$ *the unique solution of the equation* $\theta = (\ln\theta)^2$ *in the interval* $(0, +\infty)$*.*

Then $t_o > 0$ and there exists a function $\widehat{\omega} : [0, t_o] \to [0, +\infty)$ satisfying the following properties:

$$\begin{aligned} &\widehat{\omega} \text{ is continuous, concave, and strictly increasing on } [0, t_o], \\ &\widehat{\omega}(t) \geq \omega(t) \text{ for each } t \in [0, t_o], \quad \widehat{\omega}(0) = 0, \quad \widehat{\omega}(t_o) = M, \\ &\text{the mapping } \ (0, t_o) \ni t \mapsto \tfrac{\widehat{\omega}(t)}{t} \in [0, +\infty) \ \text{ is non-increasing}, \\ &\text{and } \int_0^{t_o} \frac{\widehat{\omega}(t)}{t}\, dt \leq \eta M + \Big(1 + \eta + \frac{\eta(\theta_* + |\ln \theta_*|)}{\theta_* |\ln \theta_*|}\Big) \int_0^{t_o} \frac{\omega(t)}{t}\, dt. \end{aligned} \tag{14.91}$$

Proof. We start by noting that since ω is continuous at 0 and satisfies a Dini integrability condition, then necessarily ω vanishes at the origin. In turn, this forces $t_o \in (0, R]$ and $M \in (0, +\infty)$. Given that ω is continuous, we also have that $\omega(t_o) = M$. Next, extend the restriction of ω to the interval $[0, t_o]$ to a function $\bar{\omega} : [0, +\infty) \to [0, +\infty)$ by setting $\bar{\omega}(t) := M$ for every $t \geq t_o$, and take $\widetilde{\omega} : [0, +\infty) \to [0, +\infty)$ to be the concave envelope of $\bar{\omega}$, i.e.,

$$\widetilde{\omega}(t) := \sup\Big\{ \sum_{j=1}^{N} \lambda_j \bar{\omega}(t_j) : N \in \mathbb{N},\, (\lambda_j)_j \in [0,1]^N,\, \sum_{j=1}^{N} \lambda_j = 1, (t_j)_j \in [0, +\infty)^N,\, \sum_{j=1}^{N} \lambda_j t_j = t \Big\} \tag{14.92}$$

for each $t \in [0, +\infty)$. Then (cf., e.g., the discussion in [83, pp. 35-57]), $\widetilde{\omega}$ is the smallest concave function which is pointwise $\geq \bar{\omega}$, that is,

$$\widetilde{\omega} = \inf\big\{ \psi \,:\, \psi \geq \bar{\omega} \ \text{ on } \ [0, +\infty), \ \text{ and } \ \psi \ \text{ concave on } \overline{\mathbb{R}}_+ \big\}. \tag{14.93}$$

In particular, $\widetilde{\omega}$ is concave on $[0, +\infty)$, hence continuous on $(0, +\infty)$. Also (as seen from (14.92)), we have

$$\widetilde{\omega}(0) = \omega(0) = 0 \ \text{ and } \ \widetilde{\omega}(t) = M \ \text{ for every } \ t \geq t_o. \tag{14.94}$$

Moreover, since $\widetilde{\omega}, \omega$ are continuous on $(0, R)$, formula (14.93) also entails that

$$\forall\, t \in (0, t_o) \ \text{ with } \ \widetilde{\omega}(t) > \omega(t) \Longrightarrow \begin{cases} \exists\, J \text{ open subinterval of } (0, R) \text{ so that } t \in J \\ \text{and such that } \widetilde{\omega} \text{ is an affine function on } J. \end{cases} \tag{14.95}$$

To proceed, from the fact that $\widetilde{\omega}$ and $\bar{\omega}$ are continuous on $(0, +\infty)$ and (14.94) we deduce that

$$W := \big\{ t \in (0, +\infty) : \widetilde{\omega}(t) > \bar{\omega}(t) \big\} \ \text{ is an open subset of } \ (0, t_o). \tag{14.96}$$

If W is empty, it follows that $\widetilde{\omega}(t) = \bar{\omega}(t)$ for every $t \in (0, +\infty)$, hence ω itself is concave on $(0, t_o)$ As such, we simply take $\widehat{\omega} := \omega\big|_{[0,t_o]}$ and the desired conclusion follows. There remains to study the case when the set W from (14.96) is nonempty. In this scenario, W may be written as the union of an at most countable family of mutually disjoint open intervals (which are precisely the connected components of W), say

$$W = \bigcup_{i \in I} J_i, \quad \text{where } \; J_i := (\alpha_i, \beta_i), \quad 0 \leq \alpha_i < \beta_i \leq t_o \; \text{ for each } i \in I. \tag{14.97}$$

Let us also observe that since both $\widetilde{\omega}$ and ω are continuous on $(0, R)$, from (14.94) and (14.96) we may conclude that $\widetilde{\omega}(t) = \omega(t)$ for each $t \in \partial W$. Given the nature of the decomposition of W in (14.96), this ensures that

$$\begin{aligned} &\widetilde{\omega}(t) > \omega(t) \;\text{ whenever } i \in I \text{ and } t \in (\alpha_i, \beta_i), \\ &\widetilde{\omega}(\alpha_i) = \omega(\alpha_i) \;\text{ and }\; \widetilde{\omega}(\beta_i) = \omega(\beta_i) \;\text{ for each } i \in I. \end{aligned} \tag{14.98}$$

Moreover, based on this and (14.95) we arrive at the conclusion that

$$\widetilde{\omega}(t) = \frac{t - \alpha_i}{\beta_i - \alpha_i}\big(\omega(\beta_i) - \omega(\alpha_i)\big) + \omega(\alpha_i), \;\text{ if } i \in I \text{ and } t \in [\alpha_i, \beta_i]. \tag{14.99}$$

For further use, let us point out that (14.99) readily entails

$$\widetilde{\omega}(t) \leq \omega(\alpha_i) + \frac{\omega(\beta_i)}{\beta_i} t \qquad \text{if } i \in I \text{ and } t \in [\alpha_i, \beta_i], \tag{14.100}$$

since both functions involved are affine on the interval (α_i, β_i) and the inequality is trivially verified at endpoints. Going further, fix $\theta \in (0, 1)$ and partition the (at most countable) set of indices I (from (14.97)) into the following two subclasses:

$$I_1 := \big\{i \in I : \alpha_i > \theta\beta_i\big\}, \qquad I_2 := \big\{i \in I : \alpha_i \leq \theta\beta_i\big\}. \tag{14.101}$$

Now, the fact that $\widetilde{\omega}$ is concave entails $\widetilde{\omega}\big(\lambda t_1 + (1-\lambda)t_2\big) \geq \lambda\widetilde{\omega}(t_1) + (1-\lambda)\widetilde{\omega}(t_2)$ for all $\lambda \in [0, 1]$ and $t_1, t_2 \in [0, +\infty)$. Pick now two numbers $t'' \geq t' > 0$ and specialize the earlier inequality to the case when $\lambda := t'/t''$, $t_1 := t''$ and $t_2 := 0$ (recall that $\widetilde{\omega}$ vanishes at the origin). This yields $\widetilde{\omega}(t') \geq (t'/t'')\widetilde{\omega}(t'')$, from which we may ultimately conclude that

$$\text{the mapping } \ (0,+\infty) \ni t \mapsto \frac{\widetilde{\omega}(t)}{t} \in [0,+\infty) \ \text{ is non-increasing.} \tag{14.102}$$

For each fixed $i \in I_1$, we necessarily have $\alpha_i > 0$. Keeping this in mind, we may then estimate

$$\begin{aligned} \int_{\alpha_i}^{\beta_i} \frac{\widetilde{\omega}(t)}{t}\,dt &\leq \frac{\widetilde{\omega}(\alpha_i)}{\alpha_i}(\beta_i - \alpha_i) = \frac{\omega(\alpha_i)}{\alpha_i}(\beta_i - \alpha_i) \\ &\leq \eta(\beta_i - \alpha_i)\frac{\beta_i}{\alpha_i}\Big(\inf_{t\in(\alpha_i,\beta_i)} \frac{\omega(t)}{t}\Big) \leq \eta\theta^{-1}\int_{\alpha_i}^{\beta_i} \frac{\omega(t)}{t}\,dt, \end{aligned} \tag{14.103}$$

thanks to (14.102), (14.89), and (14.101). On the other hand, when $i \in I_2$ we may write

$$\begin{aligned} \int_{\alpha_i}^{\beta_i} \frac{\widetilde{\omega}(t)}{t}\,dt &= \int_{\alpha_i}^{\beta_i} (\widetilde{\omega}(t) - \omega(\alpha_i))\,\frac{dt}{t} + \int_{\alpha_i}^{\beta_i} \omega(\alpha_i)\,\frac{dt}{t} \\ &\leq \int_{\alpha_i}^{\beta_i} \frac{\omega(\beta_i)}{\beta_i}\,dt + \eta\int_{\alpha_i}^{\beta_i} \omega(t)\,\frac{dt}{t} \leq \omega(\beta_i) + \eta\int_{\alpha_i}^{\beta_i} \frac{\omega(t)}{t}\,dt \\ &\leq \frac{\eta}{|\ln\theta|}\int_{\beta_i}^{\beta_i/\theta} \frac{\bar{\omega}(t)}{t}\,dt + \eta\int_{\alpha_i}^{\beta_i} \frac{\omega(t)}{t}\,dt, \end{aligned} \tag{14.104}$$

by (14.100), (14.89) and the definition of $\bar{\omega}$. At this stage, we proceed to estimate

$$\begin{aligned} \int_0^{t_o} \frac{\widetilde{\omega}(t)}{t}\,dt &= \int_W \frac{\widetilde{\omega}(t)}{t}\,dt + \int_{(0,t_o)\setminus W} \frac{\widetilde{\omega}(t)}{t}\,dt = \sum_{i\in I}\int_{J_i} \frac{\widetilde{\omega}(t)}{t}\,dt + \int_{(0,t_o)\setminus W} \frac{\omega(t)}{t}\,dt \\ &\leq \sum_{i\in I_1}\int_{J_i} \frac{\widetilde{\omega}(t)}{t}\,dt + \sum_{i\in I_2}\int_{J_i} \frac{\widetilde{\omega}(t)}{t}\,dt + \int_0^{t_o} \frac{\omega(t)}{t}\,dt. \end{aligned} \tag{14.105}$$

Note that (14.103) gives

$$\sum_{i\in I_1}\int_{J_i} \frac{\widetilde{\omega}(t)}{t}\,dt \leq \eta\theta^{-1}\sum_{i\in I_1}\int_{J_i} \frac{\omega(t)}{t}\,dt \leq \eta\theta^{-1}\int_0^{t_o} \frac{\omega(t)}{t}\,dt. \tag{14.106}$$

We continue by observing that

$$\forall\, i,j \in I_2 \ \text{ with } \ i \neq j \implies (\beta_i, \beta_i/\theta) \cap (\beta_j, \beta_j/\theta) = \varnothing. \tag{14.107}$$

To justify this, fix two different indices $i,j \in I_2$ and, without loss of generality, assume that $\beta_i < \beta_j$. Since (α_i,β_i) and (α_j,β_j) are disjoint connected components of W, it follows that $\beta_i \notin (\alpha_j,\beta_j)$.

Hence, $\beta_i < \alpha_j \le \theta\beta_j$ given that $j \in I_2$, which shows that $\beta_i/\theta < \beta_j$. With this in hand, (14.107) readily follows. Having established (14.107), we next invoke (14.104) in order to estimate

$$\begin{aligned} \sum_{i\in I_2}\int_{J_i}\frac{\widetilde{\omega}(t)}{t}\,dt &= \sum_{i\in I_2}\int_{\alpha_i}^{\beta_i}\frac{\widetilde{\omega}(t)}{t}\,dt \le \frac{\eta}{|\ln\theta|}\sum_{i\in I_2}\int_{\beta_i}^{\beta_i/\theta}\frac{\bar{\omega}(t)}{t}\,dt + \eta\sum_{i\in I_2}\int_{\alpha_i}^{\beta_i}\frac{\omega(t)}{t}\,dt \\ &\le \frac{\eta}{|\ln\theta|}\int_0^{t_o/\theta}\frac{\bar{\omega}(t)}{t}\,dt + \eta\int_0^{t_o}\frac{\omega(t)}{t}\,dt. \end{aligned} \tag{14.108}$$

In concert, (14.105), (14.106) and (14.108) yield

$$\begin{aligned} \int_0^{t_o}\frac{\widetilde{\omega}(t)}{t}\,dt &\le \left(1+\eta+\eta\theta^{-1}\right)\int_0^{t_o}\frac{\omega(t)}{t}\,dt + \frac{\eta}{|\ln\theta|}\int_0^{t_o/\theta}\frac{\bar{\omega}(t)}{t}\,dt \\ &= \left(1+\eta+\eta\theta^{-1}+\frac{\eta}{|\ln\theta|}\right)\int_0^{t_o}\frac{\omega(t)}{t}\,dt + \eta M. \end{aligned} \tag{14.109}$$

Finally, minimizing the right-most hand side of (14.109) over all $\theta \in (0,1)$ gives

$$\int_0^{t_o}\frac{\widetilde{\omega}(t)}{t}\,dt \le \left(1+\eta+\eta\,\theta_*^{-1}+\frac{\eta}{|\ln\theta_*|}\right)\int_0^{t_o}\frac{\omega(t)}{t}\,dt + \eta M. \tag{14.110}$$

At this point, much of the ground work ensuring that $\widehat{\omega} := \widetilde{\omega}\big|_{[0,t_o]}$ satisfies the properties listed in (14.91) has been done. Two items which are yet to be settled are as follows. First, formula (14.92) shows that $\widehat{\omega}(t) < M$ for $t \in (0,t_o)$. Hence, if $0 \le t_1 < t_2 \le t_o$ and if $\lambda := (t_o - t_2)/(t_o - t_1) \in [0,1)$ then, given that $\widehat{\omega}$ is concave, we obtain $\widehat{\omega}(t_2) \ge \lambda\widehat{\omega}(t_1) + (1-\lambda)M > \omega(t_1)$. Consequently, $\widehat{\omega}$ is strictly increasing on $[0,t_o]$. Second, the continuity of $\widehat{\omega}$ at 0 is a consequence of the fact that this function is continuous and increasing on $(0,t_o)$ and satisfies a Dini condition. This concludes the proof of the proposition. □

We are now prepared to present a consequence of Theorem 14.3 in which we impose a more streamlined set of conditions on the shape function (compare (14.111) with (14.18)-(14.19)). In turn, Theorem 14.5 below readily implies Theorem 1.6 stated in § 1.

Theorem 14.5. *Let Ω be a nonempty, proper, open subset of $\mathbb{R}^n$ and assume that $x_0 \in \partial\Omega$ is a point with the property that Ω satisfies an interior pseudo-ball condition at x_0. Concretely, assume* *that* (14.17) *holds for some parameters* $a, b, R \in (0, +\infty)$*, direction vector* $h = (h_1, ..., h_n) \in S^{n-1}$*, and a shape function* $\omega : [0, R] \to [0, +\infty)$ *with the property that*

$$\begin{aligned} &\omega \text{ is continuous, positive and quasi-increasing} \\ &\qquad \text{on } (0, R], \text{ and } \int_0^R \frac{\omega(t)}{t}\, dt < +\infty. \end{aligned} \tag{14.111}$$

Also, consider a non-divergence form, second-order, differential operator L *which is semi-elliptic and non-degenerate along* h *(as in* (14.20)-(14.21)*) and whose coefficients satisfy*

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{\sum\limits_{i=1}^{n} a^{ii}(x)}{\sum\limits_{i,j=1}^{n} a^{ij}(x)h_i h_j} < +\infty, \tag{14.112}$$

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{|x-x_0|\Big(\sum\limits_{i=1}^{n} \max\big\{0\,,\, -b^i(x)\big\}\Big)}{\sum\limits_{i,j=1}^{n} a^{ij}(x)h_i h_j} < +\infty, \tag{14.113}$$

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{\max\Big\{0\,,\, \sum\limits_{i=1}^{n} b^i(x)h_i\Big\}}{\frac{\omega(|x-x_0|)}{|x-x_0|}\Big(\sum\limits_{i,j=1}^{n} a^{ij}(x)h_i h_j\Big)} < +\infty. \tag{14.114}$$

Finally, fix a vector $\vec{\ell} \in S^{n-1}$ *for which* $\vec{\ell}\cdot h > 0$*, and suppose that a function* $u \in \mathscr{C}^0(\Omega \cup \{x_0\}) \cap \mathscr{C}^2(\Omega)$ *satisfies*

$$(Lu)(x) \geq 0 \quad \text{and} \quad u(x_0) < u(x) \quad \text{for each} \quad x \in \Omega. \tag{14.115}$$

Then $\vec{\ell}$ *points inside* Ω *at* x_0*, and there exists a constant* $\kappa > 0$ *(which depends only on the quantities in* (14.29)*) with the property that*

$$(D^{(inf)}_{\vec{\ell}} u)(x_0) \geq \kappa. \tag{14.116}$$

Proof. It may be readily verified that ω continuously vanishes at the origin given that ω satisfies a Dini integrability condition, is continuous and nonnegative on $[0, R]$, as well as quasi-increasing on $(0, R]$. Now, if $\widehat{\omega}$ is associated with the original shape function ω as in Proposition 14.4, properties (14.91) hold. In particular, $\widehat{\omega} \geq \omega$ near the origin and, hence, $\mathscr{G}^{\widehat{\omega}}_{a,b}(x_0, h) \subseteq \mathscr{G}^{\omega}_{a,b}(x_0, h) \subseteq \Omega$. Also, (14.112)-(14.114) imply the versions of (14.23)-(14.24) written with both ω and $\widetilde{\omega}$ replaced by $\widehat{\omega}$.

Then Theorem 14.3 applies, with both ω and $\widetilde{\omega}$ in the original statement replaced by $\widehat{\omega}$. From this, the desired conclusion follows. $\square$

Chapter 15

Sharpness of the Boundary Point Principle Formulated in Theorem 14.3

As mentioned previously, Theorem 14.3 is sharp, and here the goal is to make this precise through a series of counterexamples presented as remarks.

Remark 15.1. *The strict inequality in* (14.27) *is obviously necessary, since otherwise any constant function would serve as a counterexample.*

Remark 15.2. *In the context of Theorem 14.3, the nondegeneracy of L along the direction vector h of the pseudo-ball $\mathscr{G}^{\omega}_{a,b}(x_0, h)$ is a necessary condition. A simple counterexample is obtained by taking $n \geq 2$, $\Omega := \mathbb{R}^n_+$, $x_0 := (0, ..., 0) \in \mathbb{R}^n$, $\vec{\ell} := \mathbf{e}_n$, $L := -\partial^2/\partial x_1^2$ and $u(x_1, ..., x_n) := x_n^2$.*

Remark 15.3. *The discussion in § 1 pertaining to* (1.34)-(1.39) *shows that both condition* (14.23) *and condition* (14.24) *in Theorem 14.3 are necessary.*

Remark 15.4. *Fix $\alpha \in (1, 2)$ and in the two dimensional setting consider*

$$\begin{aligned}
&\Omega := \{(x, y) \in \mathbb{R}^2 : y > (x^2)^{1/\alpha}\}, \\
&L := -\partial_x^2 - \frac{2}{\alpha(\alpha+1)} y^{2-\alpha} \partial_y^2 \quad \text{in } \Omega, \qquad (15.1) \\
&u(x, y) := y^{1+\alpha} - x^2 y, \qquad \forall\, (x, y) \in \Omega.
\end{aligned}$$

Then $u \in \mathscr{C}^2(\overline{\Omega})$ satisfies $u(0) = 0$, $u > 0$ in Ω, $Lu = 0$ in Ω and $(\nabla u)(0) = 0$. Thus, (14.30) *fails in this case, even though* Ω *satisfies a pseudo-ball condition at the origin, with shape function* $\omega(t) := t^{(2/\alpha)-1}$ *satisfying* (14.18)-(14.19)*, and* L *is (non-uniformly) elliptic in* Ω *and homogeneous (i.e.,* L *has no lower order terms). Here, the breakdown is caused by the failure of condition* (14.23) *for a function* $\widetilde{\omega}$ *as in* (14.24)*. Indeed, since* $x^2 + y^2 \leq cy^\alpha$ *in* Ω*,* (14.23) *would imply* $\widetilde{\omega}(y)/y \geq c/y$ *for all* $y > 0$ *small, in violation of Dini's integrability condition for* $\widetilde{\omega}$*. Moreover, varying the parameter* $\alpha \in (1,2)$*, this counterexample shows that for any fixed* $\varepsilon > 0$ *condition* (14.23) *may not be relaxed to*

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{|x-x_0|^{\varepsilon}\Big(\sum\limits_{i=1}^{n} a^{ii}(x)\Big)}{\sum\limits_{i,j=1}^{n} a^{ij}(x)h_i h_j} < +\infty. \tag{15.2}$$

Remark 15.5. *Here the goal is to show that the conclusion* (14.30) *of Theorem 14.3 may be violated if condition* (14.24) *fails to be satisfied for some* $\widetilde{\omega}$ *as in* (14.22) *(even though* (14.17)-(14.23) *do hold for some* $\widetilde{\omega}$ *as in* (14.22)*). We start by making the general observation that if* Ω *is an arbitrary open set and if* $u \in \mathscr{C}^2(\Omega)$ *is any real-valued function without critical points in* Ω *then, obviously,*

$$-\Delta u + \Big(\frac{\Delta u}{|\nabla u|^2}\nabla u\Big)\cdot \nabla u = 0 \quad \text{ in } \ \Omega. \tag{15.3}$$

This tautology may be interpreted as the statement that u *is a null-solution of the second-order differential operator*

$$L := -\Delta + \vec{b}\cdot\nabla, \qquad \text{where} \qquad \vec{b} := \frac{\Delta u}{|\nabla u|^2}\nabla u \quad \text{in } \ \Omega. \tag{15.4}$$

Let us now specialize these general considerations to the case when (in the two-dimensional setting)

$$\Omega := \mathbb{R}^2_+ \cap B(\mathbf{0}, e^{-1}) \ \text{ and } \ u(x,y) := y\big[-\ln\sqrt{x^2+y^2}\,\big]^{-\varepsilon} \quad \forall\,(x,y)\in\Omega, \tag{15.5}$$

where $\mathbf{0} := (0,0)$ *is the origin in* $\mathbb{R}^2$*, and* $\varepsilon > 0$ *is a fixed, small number. Clearly,* (14.17)-(14.23) *do hold and* Ω *does satisfy an interior pseudo-ball condition at* $\mathbf{0} \in \partial\Omega$*, if we take* $\omega(t) := \widetilde{\omega}(t) := t^\alpha$ *for some arbitrary, fixed* $\alpha \in (0,1)$*. Note that such a choice guarantees that both* (14.18)-(14.19) *and*

(14.22) *are satisfied. Going further, a direct computation in polar coordinates* (r,θ) *shows that*

$$(\nabla u)(r,\theta) = \Big(\varepsilon \sin\theta\cos\theta(-\ln r)^{-\varepsilon-1}\,,\,(-\ln r)^{-\varepsilon-1}(\varepsilon\sin^2\theta - \ln r)\Big), \tag{15.6}$$

so choosing ε *small enough ensures that* u *does not have critical points in* Ω. *Assuming that this is the case, the drift coefficients* $\vec{b} = (b^1, b^2)$ *of the operator* L *associated with this function may be expressed in polar coordinates* (r,θ) *as*

$$b^1(r,\theta) = \frac{\varepsilon^2\sin^2\theta\cos\theta(2\ln r - 1 - \varepsilon)}{r(\ln r)\big[\varepsilon\sin^2\theta(\varepsilon - 2\ln r) + (\ln r)^2\big]}, \tag{15.7}$$

$$b^2(r,\theta) = \frac{\varepsilon\sin\theta(2\ln r - 1 - \varepsilon)(\varepsilon\sin^2\theta - \ln r)}{r(\ln r)\big[\varepsilon\sin^2\theta(\varepsilon - 2\ln r) + (\ln r)^2\big]}. \tag{15.8}$$

It is then clear from (15.5) *that* $u > 0$ *in* Ω, $u \in \mathscr{C}^2(\Omega)$, *and that* u *may be continuously extended to* $\Omega \cup \{\mathbf{0}\}$ *by setting* $u(\mathbf{0}) := 0$. *Furthermore, as is readily seen from* (15.6), *the fact that* $\varepsilon > 0$ *forces* $\lim\limits_{y\to 0^+}(\partial_y u)(x,y) = 0$, *uniformly in* x. *As a result,* $(D^{(inf)}_{\mathbf{e}_2}u)(\mathbf{0}) = 0$ *which shows that the conclusion in Theorem 14.3 fails. The reason for this failure is the fact that condition* (14.24) *does not hold in the current situation for any choice of* $\widetilde{\omega}$ *as in* (14.22). *Indeed, if* (14.24) *were to hold, it would then be possible to find a constant* $c > 0$ *with the property that*

$$\frac{\widetilde{\omega}(r)}{r} \geq c\,\max\{0, b^2(r, \tfrac{\pi}{2})\} \geq \frac{c_\varepsilon}{r(-\ln r)} \qquad \textit{for all } r > 0 \textit{ small,} \tag{15.9}$$

where $c_\varepsilon > 0$ *depends only on* ε. *However, this would then imply that* $\widetilde{\omega}$ *fails to satisfy Dini's integrability condition since* $\int_0^{e^{-1}} \frac{1}{r(-\ln r)}\,dr = \int_1^{+\infty} s^{-1}\,ds = +\infty$ *(after making the change of variables* $r = e^{-s}$*).*

The above discussion also shows that condition (14.24) *may not be weakened to*

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{|\vec{b}(x)|\big|\ln|x-x_0|\big|^{-\delta}}{\frac{\widetilde{\omega}((x-x_0)\cdot h)}{(x-x_0)\cdot h}\Big(\sum\limits_{i,j=1}^{n} a^{ij}(x)h_i h_j\Big)} < +\infty, \qquad \textit{for some } \ \delta > 0. \tag{15.10}$$

Indeed, in the case of (15.4)-(15.5), *such a weakened condition would be satisfied for any given* $\delta > 0$ *by taking, in the notation introduced in* (1.18), $\widetilde{\omega} := \omega_{0,-1-\delta}$ *i.e.,* $\widetilde{\omega}(t) = |\ln t|^{-1-\delta}$. *However, as* *already noted, the conclusion in Theorem 14.3 fails for* (15.4)-(15.5).

The same type of counterexample may be easily adapted to the higher-dimensional setting, taking $\Omega := \mathbb{R}^n_+ \cap B(0, e^{-1})$ *and* $u(x) := x_n(-\ln|x|)^{-\varepsilon}$ *in place of* (15.5). *In this case, the drift coefficients continue to exhibit the same type of singularity at the origin as* (15.7)-(15.8). *In particular, we have*

$$\vec{b} : \Omega \longrightarrow \mathbb{R}^n, \qquad |\vec{b}(x)| = O\Big(\frac{1}{|x|\big|\ln|x|\big|}\Big) \quad \text{as} \quad |x| \to 0, \tag{15.11}$$

which shows that

$$\vec{b} \in L^n(\Omega). \tag{15.12}$$

This should be compared with the classical Aleksandrov-Bakel'man-Pucci theorem which asserts that the Weak Maximum Principle holds for uniformly elliptic operators in open subsets of $\mathbb{R}^n$ *whose drift coefficients are locally in* L^n. *In this light, the significance of* (15.12) *is that, in contrast with the Aleksandrov-Bakel'man-Pucci Weak Maximum Principle, the Boundary Point Principle may fail even though the drift coefficients are in* L^n. *See also [85, Example 1.12], [85, Example 4.1], [73, Remark 3] in this regard.*

Remark 15.6. *Here we present another example for which the same type of conclusions (pertaining the singularity of the drift coefficients) as in Remark 15.5 may be inferred. Specifically, consider the domain* $\Omega := \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + y^2 < 1\} \subseteq \mathbb{R}^2$ *and define the function* $u : \overline{\Omega} \to \mathbb{R}$ *by setting*

$$u(x, y) := x\, e^{-\sqrt{-\ln[(x^2+y^2)/4]}} \quad \text{for each } (x, y) \in \overline{\Omega} \setminus \{\mathbf{0}\}, \text{ and } u(\mathbf{0}) := 0, \tag{15.13}$$

where, as before, $\mathbf{0}$ *denotes the origin in* $\mathbb{R}^2$. *Then it is not difficult to check that* $u \in \mathscr{C}^1\big(\overline{\Omega}\big) \cap \mathscr{C}^\infty(\Omega)$, $u > 0$ *in* Ω *and* $(\nabla u)(\mathbf{0}) = 0$. *Furthermore, as noted in [29, p. 169], the function* u *is satisfies the divergence-form, elliptic, second-order equation*

$$\partial_x\big(a\, \partial_x u + b\, \partial_y u\big) + \partial_y\big(b\, \partial_x u + c\, \partial_y u\big) = 0 \quad \text{in} \quad \Omega, \tag{15.14}$$

where the coefficients $a, b, c \in \mathscr{C}^0(\overline{\Omega}) \cap \mathscr{C}^\infty(\Omega)$ *are defined as follows:*

$$a := \frac{1}{\mu} + \frac{y^2(\mu^2-1)}{(x^2+y^2)\mu}, \;\; b := \frac{xy(1-\mu^2)}{(x^2+y^2)\mu}, \;\; c := \frac{1}{\mu} + \frac{x^2(\mu^2-1)}{(x^2+y^2)\mu} \;\; \text{in } \overline{\Omega} \setminus \{\mathbf{0}\}, \;\; \text{where}$$
$$\mu := 1 + \big(2\sqrt{-\ln[(x^2+y^2)/4]}\,\big)^{-1}, \text{ and } a(\mathbf{0}) := 1,\; b(\mathbf{0}) := 0,\; c(\mathbf{0}) := 1. \tag{15.15}$$

Taking advantage of the differentiability of these coefficients, we may convert (15.14) *into the uniformly elliptic, non-divergence form, second-order equation* $Lu = 0$ *in* Ω*, where*

$$L := -\sum_{i,j=1}^{2} a^{ij}\partial_i\partial_j + \sum_{i=1}^{2} b^i\partial_i \;\; \text{with } a^{11} := -a,\; a^{22} := -c,\; a^{12} := a^{21} := -c,$$
$$\text{and with } b^1 := -\partial_x a - \partial_y b, \quad b^2 := -\partial_y c - \partial_x b \;\; \text{in } \Omega. \tag{15.16}$$

Then the top-coefficients of L *are bounded in* Ω*, while the drift coefficients exhibit the following type of behavior near the origin:*

$$b^i(x,y) \;\; \text{blows up at } \mathbf{0} \text{ like } \frac{1}{\sqrt{x^2+y^2}(-\ln(x^2+y^2))^{3/2}}, \qquad i = 1, 2. \tag{15.17}$$

Then (compare with (15.11)*), the same type of conclusions as in Remark 15.5 may be drawn in this case as well.*

Remark 15.7. *The point of the next example is to show that if the Dini condition on* $\widetilde{\omega}$ *is allowed to fail (while all the other hypotheses are retained) then* (14.30) *is no longer expected to hold, even for such simple differential operators as* $L := -\Delta$*. To see that this is the case, denote by* $\mathbf{0}$ *the origin of* $\mathbb{R}^2$ *and consider the two-dimensional domain*

$$\Omega := \Big\{(x,y) \in B\big(\mathbf{0}, e^{-1}\big) \setminus \{\mathbf{0}\} : \sqrt{x^2+y^2} + y\ln\sqrt{x^2+y^2} < 0\Big\} \subseteq \mathbb{R}^2. \tag{15.18}$$

Then Ω *satisfies an interior pseudo-ball condition at* $\mathbf{0} \in \partial\Omega$ *given that, in fact,*

$$\Omega = \mathscr{G}_{1,1}^{\omega_{0,-1}}(\mathbf{0}, \mathbf{e}_2) \tag{15.19}$$

where the shape function $\omega_{0,-1}$ *is as in* (1.18)*; that is,* $\omega_{0,-1}(t) = \frac{-1}{\ln t}$ *if* $t \in (0, \frac{1}{e}]$ *and* $\omega_{0,-1}(0) = 0$.

Next, pick $\varepsilon \in (0, \frac{1}{2})$ *and define* $u : \Omega \cup \{\mathbf{0}\} \to \mathbb{R}$ *by setting for each* $(x,y) \in \Omega \cup \{\mathbf{0}\}$,

$$u(x,y) := \begin{cases} \Big(y + \dfrac{\sqrt{x^2+y^2}}{\ln\sqrt{x^2+y^2}}\Big)\big(-\ln\sqrt{x^2+y^2}\big)^{-\varepsilon} & \text{if } (x,y) \neq \mathbf{0}, \\ 0 & \text{if } (x,y) = \mathbf{0}. \end{cases} \tag{15.20}$$

Then, clearly, $u \in \mathscr{C}^0(\Omega \cup \{\mathbf{0}\}) \cap \mathscr{C}^2(\Omega)$ and $u(\mathbf{0}) < u(x,y)$ for every $(x,y) \in \Omega$. Working in polar coordinates (r,θ), an elementary calculation (recall that here $L := -\Delta$) shows that, in Ω,

$$(Lu)(r,\theta) = \frac{1}{r\,(-\ln r)^{\varepsilon+3}}\Big\{(1-2\varepsilon\sin\theta)(\ln r)^2 + (\varepsilon+1)(\varepsilon\sin\theta-2)\ln r + (\varepsilon+1)(\varepsilon+2)\Big\}. \tag{15.21}$$

Since the squared logarithm in the curly brackets above has a positive coefficient given that $\varepsilon \in (0,\frac{1}{2})$, we infer that $(Lu)(x,y) \geq 0$ at each point (x,y) in Ω. On the other hand, a direct calculation gives that, for each (x,y) in Ω,

$$\begin{aligned}(\partial_y u)(x,y) \;=\; & \Big\{1 + \frac{2y}{\sqrt{x^2+y^2}}\frac{1}{\ln(x^2+y^2)} - \frac{4y}{\sqrt{x^2+y^2}}\frac{1}{\big(\ln(x^2+y^2)\big)^2}\Big\}\big(-\ln(x^2+y^2)\big)^{-\varepsilon} \\ & + \varepsilon\Big\{\frac{2y^2}{x^2+y^2} + \frac{4y}{\sqrt{x^2+y^2}}\frac{1}{\ln(x^2+y^2)}\Big\}\big(-\ln(x^2+y^2)\big)^{-\varepsilon-1}.\end{aligned} \tag{15.22}$$

Since the two expressions in curly brackets are bounded and $\varepsilon > 0$, it follows that $\lim_{y\to 0^+}(\partial_y u)(x,y) = 0$, uniformly in x. Thus, ultimately, $(D^{(inf)}_{\mathbf{e}_2}u)(\mathbf{0}) = 0$, i.e., the lower directional derivative of u at $\mathbf{0}$ along $\mathbf{e}_2$ is in fact null. As such, the conclusion in Theorem 14.3 fails. The source of this breakdown is the fact that for any continuous function $\omega : [0,R] \to [0,+\infty)$ and any $a,b > 0$ with the property that $\mathscr{G}^{\omega}_{a,b}(\mathbf{0},\mathbf{e}_2) \subseteq \Omega$, from (15.19) we deduce that $\omega(t) \geq a^{-1}\omega_{0,-1}(t)$ for each $t > 0$ sufficiently small. Granted this and given that $\int_0^{1/e}\frac{\omega_{0,-1}(t)}{t}\,dt = +\infty$, we conclude that ω necessarily fails to satisfy Dini's integrability condition. In concert with (14.32), this ultimately shows that $\widetilde{\omega}$ fails to satisfy Dini's integrability condition.

Remark 15.8. *There exists a bounded, convex domain, which is globally of class $\mathscr{C}^1$ as well as of class $\mathscr{C}^\infty$ near all but one of its boundary points, and with the property that the conclusion in the Boundary Point Principle in Theorem 14.3 fails, even for such simple differential operators as $L := -\Delta$.*

Indeed, it suffices to show that the two-dimensional domain Ω introduced in (15.18) is convex and of class $\mathscr{C}^1$ near the origin $\mathbf{0}$ of $\mathbb{R}^2$, and of class $\mathscr{C}^\infty$ near each point on $\partial\Omega \setminus \{\mathbf{0}\}$ near the origin.

With this goal in mind, we seek a representation of $\partial\Omega$ near $\mathbf{0}$ as the graph of some real-valued function $f \in \mathscr{C}^1\big((-r,r)\big)$, for some small $r > 0$, which is $\mathscr{C}^\infty$ on $(-r,r) \setminus \{0\}$, vanishes at 0, and such that $f''(x) > 0$ for every $x \in (-r,r) \setminus \{0\}$. To get started, define $F : \mathbb{R}^2 \setminus \{\mathbf{0}\} \longrightarrow \mathbb{R}$, by

$$F(x,y) := \sqrt{x^2+y^2} + y \ln\sqrt{x^2+y^2}, \quad \forall\,(x,y) \in \mathbb{R}^2 \setminus \{\mathbf{0}\}, \tag{15.23}$$

and note that a point (x,y) near the origin in $\mathbb{R}^2$ belongs to $\partial\Omega$ if and only if $F(x,y) = 0$. Also, fix a number $r \in (0, e^{-10})$ and observe that for each $x \in (-r,r) \setminus \{0\}$ we have $F(x,0) = |x| > 0$ and $F(x, \sqrt{4r^2 - x^2}) = 2r + \sqrt{4r^2-x^2}\ln(2r) < 0$. Moreover, since

$$\partial_y F(x,y) = \frac{y}{\sqrt{x^2+y^2}} + \ln\sqrt{x^2+y^2} + \frac{y^2}{x^2+y^2} < 0 \quad \text{in } B(\mathbf{0}, r), \tag{15.24}$$

it follows that $F(x,\cdot)$ is strictly decreasing near 0. This shows that if for each fixed $x \in (-r,r) \setminus \{0\}$ we define $f(x)$ to be the unique number $y \in (0, \sqrt{4r^2-x^2})$ such that $F(x,y) = 0$, and also set $f(0) := 0$, then the upper-graph of f coincides with Ω near $\mathbf{0}$ and $F(x, f(x)) = 0$ for every $x \in (-r,r)$. Furthermore, since f is bounded and

$$\sqrt{x^2+f(x)^2} + f(x)\ln\sqrt{x^2+f(x)^2} = 0, \qquad \forall\, x \in (-r,r) \setminus \{0\}, \tag{15.25}$$

a simple argument shows that $\lim\limits_{x\to 0} f(x) = 0$, so that $f \in \mathscr{C}^0\big((-r,r)\big)$. On the other hand, the fact that $F(x, f(x)) = 0$ for every $x \in (-r,r)$ gives, on account of the Implicit Function Theorem, that $f \in \mathscr{C}^\infty\big((-r,r)\setminus\{0\}\big)$ and, for each $x \in (-r,r) \setminus \{0\}$,

$$\begin{aligned} f'(x) &= -\frac{\dfrac{x}{\sqrt{x^2+y^2}}\dfrac{1}{\ln\sqrt{x^2+y^2}} + \dfrac{xy}{(x^2+y^2)}\dfrac{1}{\ln\sqrt{x^2+y^2}}}{\dfrac{y}{\sqrt{x^2+y^2}}\dfrac{1}{\ln\sqrt{x^2+y^2}} + 1 + \dfrac{y^2}{(x^2+y^2)}\dfrac{1}{\ln\sqrt{x^2+y^2}}} \\ &= \frac{xf(x)\big(f(x)+\sqrt{x^2+f(x)^2}\big)}{x^2\sqrt{x^2+f(x)^2} - f(x)^3}. \end{aligned} \tag{15.26}$$

The first formula above readily gives that $\lim\limits_{x\to 0} f'(x) = 0$. Based on this and the Mean Value Theorem, we arrive at the conclusion that f is differentiable at 0 and $f'(0) = 0$. Thus, ultimately, we have

$f \in \mathscr{C}^1\big((-r,r)\big) \cap \mathscr{C}^\infty\big((-r,r)\setminus\{0\}\big)$. *Going further, based on the second formula for* f' *in* (15.26) *and* (15.25)*, an involved but elementary calculation shows that for each* $x \in (-r,r)\setminus\{0\}$ *we have*

$$\begin{aligned} f''(x) \;&=\; \frac{f(x)^2(x^2+f(x)^2)}{\big(x^2\sqrt{x^2+f(x)^2}-f(x)^3\big)^3}\times \\ &\qquad \times\Big\{f(x)(2x^2+f(x)^2)\sqrt{x^2+f(x)^2}+x^4+x^2f(x)^2+f(x)^4\Big\}. \end{aligned} \tag{15.27}$$

In turn, since $x^2\sqrt{x^2+f(x)^2}-f(x)^3 = x^3\big(\sqrt{1+(f(x)/x)^2}-(f(x)/x)^3\big) > 0$ *if* $r>0$ *is small, thanks to the fact that* $f'(0)=0$*, we may conclude from* (15.27) *that* $f''(x)>0$*, as desired.*

Remark 15.9. *Here we strengthen the counterexample discussed in Remark 15.8 by showing that there exists a bounded, convex domain, which is globally of class* $\mathscr{C}^1$ *as well as of class* $\mathscr{C}^\infty$ *near all but one of its boundary points, and with the property that the conclusion in the Boundary Point Principle in Theorem 14.3 fails for* $L := -\Delta$ *even under the assumption that* u *is a null-solution in* Ω *(i.e.,* u *is a harmonic function).*

To see that this is the case, we shall work in the two-dimensional setting and, following a suggestion from [30, p. 35], for every point $(x,y)\in\mathbb{R}^2\setminus\big\{(x,0):\, x\in(-\infty,0]\cup\{1\}\big\}$ *define*

$$u(x,y) := \mathrm{Re}\Big(\frac{x+iy}{-\ln(x+ix)}\Big) = \frac{-x\ln(\sqrt{x^2+y^2})-y\,\mathrm{Arg}(x,y)}{(\ln(\sqrt{x^2+y^2}))^2+(\mathrm{Arg}(x,y))^2}, \tag{15.28}$$

where Arg : $\mathbb{R}^2\setminus\big((-\infty,0]\cup\times\{0\}\big)\to\big(-\frac{\pi}{2},\frac{\pi}{2}\big)$*, defined as*

$$\mathrm{Arg}(x,y) := \begin{cases} \arctan\Big(\frac{y}{x}\Big), & \text{if } x\geq 0,\ y\in\mathbb{R},\ (x,y)\neq(0,0), \\ \pi+\arctan\Big(\frac{y}{x}\Big), & \text{if } x<0,\ y>0, \\ -\pi+\arctan\Big(\frac{y}{x}\Big), & \text{if } x<0,\ y<0, \end{cases} \tag{15.29}$$

is the argument of the complex number $z := x+iy\in\mathbb{C}$*. In particular,* Arg *is* $\mathscr{C}^\infty$ *on its domain, and* $\partial_x\mathrm{Arg}(x,y) = -y(x^2+y^2)^{-1}$ *and* $\partial_y\mathrm{Arg}(x,y) = x(x^2+y^2)^{-1}$ *there. Next, consider the open subset of* $\mathbb{R}^2$ *given by*

$$\Omega := \Big\{(x,y)\in\mathbb{R}^2\setminus\big\{(x,0):\, x\in(-\infty,0]\cup\{1\}\big\} :\, u(x,y)>0\Big\}. \tag{15.30}$$

Then $u \in \mathscr{C}^\infty(\Omega)$ *and is harmonic in* Ω*, since* u *is the real part of the complex-valued function* $\frac{z}{-\ln z}$*, which is analytic there. Moreover, it is clear that* u *may be continuously extended to* $\mathbf{0}$ *by setting* $u(\mathbf{0}) := 0$*. Also,* $u > 0$ *in* Ω *by design. To proceed, introduce the continuous function*

$$F(x,y) := \begin{cases} \frac{x}{2}\ln(x^2+y^2) + y\,\mathrm{Arg}(x,y) & \textit{if } (x,y) \in \mathbb{R}^2 \setminus \big((-\infty,0]\times\{0\}\big), \\ 0 & \textit{if } (x,y) = (0,0) \end{cases} \tag{15.31}$$

and note that $F \in \mathscr{C}^\infty\big(\mathbb{R}\setminus\{0\})\times\mathbb{R}\big)$*. The significance of this function stems from the fact that*

$$\Omega = \Big\{(x,y) \in \mathbb{R}^2 \setminus \big((-\infty,-1)\times\{0\}\big) : F(x,y) < 0\Big\}. \tag{15.32}$$

A careful elementary analysis of the nature of the function F *shows that there exists* $\theta \in (0, \frac{\pi}{2})$ *such that for any* $y \in (-e^{-2}\sin\theta, e^{-2}\sin\theta)\setminus\{0\}$ *the function* $F(\cdot,y) : \{x \in \mathbb{R} : (x,y) \in B(\mathbf{0}, e^{-2})\} \to \mathbb{R}$ *is continuous, strictly decreasing, and satisfies* $F(0,y) > 0$ *and* $F(\sqrt{e^{-4}-y^2}, y) < 0$*. Consequently, for each* $y \in (-e^{-2}\sin\theta, e^{-2}\sin\theta)\setminus\{0\}$ *there exists a unique number*

$$f(y) \in \big(0, \sqrt{e^{-4}-y^2}\,\big) \quad \textit{such that} \quad F(f(y),y) = 0. \tag{15.33}$$

The Implicit Function Theorem then shows that $f : (-e^{-2}\sin\theta, e^{-2}\sin\theta)\setminus\{0\} \to (0,+\infty)$ *just defined is of class* $\mathscr{C}^\infty$*. Moreover, a simple argument based on* (15.33) *gives that* $\lim\limits_{y\to 0} f(y) = 0$*. Therefore, setting* $f(0) := 0$ *extends* f *continuously to the entire interval* $(-e^{-2}\sin\theta, e^{-2}\sin\theta)$*.*

We claim that actually $f \in \mathscr{C}^1\big((-e^{-2}\sin\theta, e^{-2}\sin\theta)\big)$*. To justify this claim, we first note that, by the Implicit Function Theorem*

$$f'(y) = -\frac{(\partial_y F)(f(y),y)}{(\partial_x F)(f(y),y)} = -\frac{\dfrac{2f(y)y}{f(y)^2+y^2} + \mathrm{Arg}\,(f(y),y)}{\dfrac{1}{2}\ln(f(y)^2+y^2) + \dfrac{f(y)^2-y^2}{f(y)^2+y^2}}, \qquad y \neq 0. \tag{15.34}$$

Given that both the numerator of the fraction in the right-hand side of (15.34) *and the expression* $(f(y)^2-y^2)/(f(y)^2+y^2)$ *in the denominator are bounded, while the logarithmic factor converges to* $-\infty$ *as* $y \to 0$*, we deduce that* $\lim\limits_{y\to 0} f'(y) = 0$*. In turn, from this and the Mean Value Theorem we may then conclude that the function* f *is differentiable at* 0*,* $f'(0) = 0$ *and, moreover, that*

$f \in \mathscr{C}^1\big((-e^{-2}\sin\theta, e^{-2}\sin\theta)\big)$.

Moving on, if $U := \{(x,y) \in B(\mathbf{0}, e^{-2}) : |y| < e^{-2}\sin\theta\}$*, the manner in which the function* f *has been designed ensures that*

$$U \cap \Omega = U \cap \big\{(x,y) \in \mathbb{R}^2 : y \in (-e^{-2}\sin\theta, e^{-2}\sin\theta) \text{ and } x > f(y)\big\}. \tag{15.35}$$

The latter implies that Ω *is of class* $\mathscr{C}^1$ *near* $\mathbf{0}$ *and of class* $\mathscr{C}^\infty$ *near any point on* $\partial\Omega$ *sufficiently close to* $\mathbf{0}$*. We also claim that* Ω *is convex near* $\mathbf{0}$*. To see this, we make use of the fact that* $F(f(y), y) = 0$ *and re-write* (15.34) *in the form*

$$f'(y) = \frac{2f(y)^2 y + f(y)(f(y)^2 + y^2)\mathrm{Arg}\,(f(y), y)}{y(f(y)^2 + y^2)\mathrm{Arg}\,(f(y), y) + f(y)(y^2 - f(y)^2)}, \qquad y \neq 0. \tag{15.36}$$

Differentiating this and once more making use of the fact that $F(f(y), y) = 0$ *then yields (after a lengthy yet elementary calculation)*

$$\begin{aligned} f''(y) \;&=\; \frac{1}{\big(y(f(y)^2 + y^2)\mathrm{Arg}\,(f(y), y) + f(y)(y^2 - f(y)^2)\big)^2} \times \Big\{\Big(5f(y)^4 y\,\mathrm{Arg}\,(f(y), y) \\ &\quad+\; 2f(y)^3(f(y)^2 + y^2)\,\mathrm{Arg}\,(f(y), y)^2\Big) + 3f(y)^3(y^2 - f(y)^2) \\ &\quad-\; \frac{\big(2f(y)^2 y + f(y)(f(y)^2 + y^2)\,\mathrm{Arg}\,(f(y), y)\big)^2\big(2f(y)y\,\mathrm{Arg}\,(f(y), y) - 3f(y)^2\big)}{y(f(y)^2 + y^2)\mathrm{Arg}\,(f(y), y) + f(y)(y^2 - f(y)^2)}\Big\} \end{aligned} \tag{15.37}$$

for $y \neq 0$*. Note that* $3f(y)^3(y^2 - f(y)^2) = 3f(y)^3 y^2\big(1 - (f(y)/y)^2\big)$ *and* $\big(1 - (f(y)/y)^2\big) \to 1$ *as* $y \to 0$*. Since the last fraction in* (15.37) *may be written as* $f(y)^3 y^2\big(-\frac{\pi^2}{4} + o(1)\big)$ *as* $y \to 0$*, this analysis shows that* $f''(y) > 0$ *for all* $y \neq 0$ *sufficiently close to* 0*. The bottom line is that* Ω *is convex near* $\mathbf{0}$*.*

However, as it is easily checked from (15.28)*, the inner normal derivative of the function* u *to* $\partial\Omega$ *vanishes at the origin, so the Boundary Point Principle fails even for harmonic functions in this domain.*

A more insightful explanation is offered by the following observation. For any continuous function

ω with the property that Ω satisfies an interior pseudo-ball condition at $\mathbf{0}$ with shape function ω, we necessarily have $\sqrt{f(y)^2+y^2}\,\omega(\sqrt{f(y)^2+y^2}) \geq f(y)$ for $y>0$ small. Hence, if ω is slowly growing (say, $\omega(2t) \leq c\omega(t)$ for all $t>0$ small), then $\omega(y) \geq c\,f(y)/y$ for all $y>0$ small, for some constant $c>0$. As a consequence, if $R>0$ is small then

$$\int_0^R \frac{\omega(y)}{y}\,dy \geq \int_0^R \frac{f(y)}{y^2}\,dy = -R^{-1}f(R) + \int_0^R \frac{f'(y)}{y}\,dy, \tag{15.38}$$

after an integration by parts. However, based on formula (15.34) *and the fact that $f(y)/y \to 0$, $\mathrm{Arg}(f(y),y) \to \pi/2$ as $y \to 0$, it is not difficult to see that $f'(y)/y \geq c/(-y\ln y)$ for all $y>0$ small, where $c>0$ is a fixed constant. Hence, $\int_0^R \frac{f'(y)}{y}\,dy = +\infty$ which shows that Ω fails to satisfy an interior pseudo-ball condition at $\mathbf{0}$ with a shape function for which Dini's integrability condition holds.*

In the context of Theorem 14.3, the significance of this failure is that any function $\widetilde{\omega}$ for which (14.23) *holds will, thanks to* (14.32)*, necessarily fail to satisfy Dini's integrability condition, thus contradicting the last condition in* (14.22)*.*

The harmonic function $u(x,y) := xy$ for $x,y>0$ is a counterexample to the Boundary Point Principle for $L := -\Delta$ when Ω is the first quadrant in the two-dimensional setting. A related counterexample in an arbitrary sector in the plane is presented in [85, Example 1.6]. Compared to these, the counterexamples discussed in Remark 15.8 and Remark 15.9 are considerably stronger since they deal with open sets from the much more smaller class of $\mathscr{C}^1$ domains whose unit normal has a modulus of continuity which fails to satisfy Dini's integrability condition.

We conclude this chapter with a comment pertaining to the nature of the Boundary Point Principle proved by M. Safonov in [86, Theorem 4.3 and Remark 4.4, p. 18]. Specifically, the demands here are that L is uniformly elliptic and that a truncated circular cylinder Q which touches the boundary at x_0 may be placed inside Ω and that the drift coefficients belong to $L^q(\Omega)$ for some $q > n$. What we wish to note here is that there exist vector fields $\vec{b} = (b^1, ..., b^n)$ which satisfy (14.113)-(14.114) for some shape function ω as in (14.111) but for which

$$\vec{b} \notin \bigcup_{q>n} L^q(\Omega). \tag{15.39}$$

For example, one may take $\omega : (0, 1/e) \to (0, +\infty)$ given by $\omega(t) := (\ln t)^{-2}$ for each $t \in (0, 1/e)$, and $\vec{b} : \Omega \to \mathbb{R}$ such that

$$|\vec{b}(x)| \approx \frac{1}{|x - x_0|(\ln|x - x_0|)^2}, \qquad \text{uniformly for } x \in \Omega. \tag{15.40}$$

Chapter 16

The Strong Maximum Principle for Non-uniformly Elliptic Operators with Singular Drift

The Strong Maximum Principle (SMP) is a bedrock result in the theory of second order elliptic partial differential equations, since it enables us to derive information about solutions of differential inequalities without any explicit knowledge of the solutions themselves. In reference to the seminal work of E. Hopf in [42], J. Serrin wrote in [71, p. 9]: "*It has the beauty and elegance of a Mozart symphony, the light of a Vermeer painting. Only a fraction more than five pages in length, it still contains seminal ideas which are still fresh after 75 years.*" The traditional formulation of SMP typically requires the coefficients to be locally bounded (among other things), and here our goal is to prove a version of SMP in which this assumption is relaxed to an optimal pointwise blow-up condition. Specifically, we shall prove the following theorem.

Theorem 16.1. *Let Ω be a nonempty, connected, open subset of $\mathbb{R}^n$, and suppose that*

$$L := -\mathrm{Tr}\big(A\,\nabla^2\big) + \vec{b}\cdot\nabla = -\sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j + \sum_{i=1}^{n} b^i\partial_i \tag{16.1}$$

is a (possibly non-uniformly) elliptic second-order differential operator in non-divergence form (without a zero-order term) in Ω. Also, assume that for each $x_0 \in \Omega$ and each $\xi \in S^{n-1}$ there exists a *real-valued function* $\widetilde{\omega} = \widetilde{\omega}_{x_0,\xi}$ *satisfying*

$$\widetilde{\omega} \in \mathscr{C}^0\big([0,1]\big), \quad \widetilde{\omega}(t) > 0 \ \text{ for each } \ t \in (0,1], \quad \int_0^1 \frac{\widetilde{\omega}(t)}{t}\,dt < \infty, \tag{16.2}$$

and with the property that

$$\limsup_{\substack{(x-x_0)\cdot\xi>0 \\ x\to x_0}} \frac{\Big(\mathrm{Tr}\,A(x)\Big) + \max\Big\{0\,,\vec{b}(x)\cdot\xi\Big\} + \Big(\sum_{i=1}^{n}\max\big\{0\,,\,-b^i(x)\big\}\Big)|x-x_0|}{\frac{\widetilde{\omega}((x-x_0)\cdot\xi)}{(x-x_0)\cdot\xi}\Big((A(x)\xi)\cdot\xi\Big)} < \infty. \tag{16.3}$$

Let $u \in \mathscr{C}^2(\Omega)$ *be a function which satisfies the differential inequality* $(Lu)(x) \geq 0$ *for all* $x \in \Omega$.

Then

if u *assumes a global minimum value at some point in* Ω*, it follows that* u *is constant in* Ω. (16.4)

Remark 16.1. *We wish to emphasize that no assumption on the (Lebesgue) measurability of the coefficients* a^{ij}, b^i, *of the operator* L *is made in the statement of the above theorem.*

Proof of Theorem 16.1. The proof proceeds along the lines of the classical Hopf's Strong Maximum Principle (as presented in, e.g., [30, Theorem 3.5, p. 35]), with the Boundary Point Principle established in Theorem 14.5 replacing its weaker, more familiar, counterpart. With the goal of arriving at a contradiction, suppose that $u \in \mathscr{C}^2(\Omega)$ is a non-constant function satisfying $Lu \geq 0$ in Ω and which assumes a global minimum value $M \in \mathbb{R}$ at some point $x_* \in \Omega$. Then if $U := \{x \in \Omega :\ u(x) = M\}$, it follows that U is a nonempty, relatively closed, proper subset of the connected set Ω hence, in order to reach a contradiction, it suffices to show that U is open, i.e. that $U \setminus U^\circ = \varnothing$. To this end, reason by contradiction and assume that there exists $y \in U \setminus U^\circ$. Since Ω is open and $y \in \Omega$, one may pick $r > 0$ such that $B(y,r) \subseteq \Omega$. On the other hand, the fact that $y \in U \setminus U^\circ$ implies that $B(y, r/2)$ is not contained in U. Hence, there exists $z \in B(y, r/2) \setminus U$ and we select $x_0 \in U$ with the property that $\mathrm{dist}\,(z, U) = |z - x_0| =: R > 0$ (since U is relatively closed). In turn, such a choice forces $\mathrm{dist}\,(z, \partial\Omega) > r/2 > |y - z| \geq \mathrm{dist}\,(z, U) = R$, hence ultimately

$$B(z,R) \subseteq \Omega \setminus U \quad \text{and} \quad x_0 \in U \cap \partial B(z,R). \tag{16.5}$$

For further use, let us also note here that the fact that $x_0 \in U$ and (16.5) entail, respectively,

$$(\nabla u)(x_0) = 0 \quad \text{and} \quad x_0 \in \partial(\Omega \setminus U). \tag{16.6}$$

To proceed, define $h := R^{-1}(z - x_0) \in S^{n-1}$ and let $\widetilde{\omega} : (0,1) \to (0,+\infty)$ be the function associated with the point $x_0 \in \Omega$ and the vector $h \in S^{n-1}$ as in the statement of the theorem. On account of (16.5) it follows that the open, nonempty set $\Omega \setminus U$ satisfies a pseudo-ball condition at the point $x_0 \in \partial(\Omega \setminus U)$ with shape function $\omega(t) := t$ and direction vector $h = R^{-1}(z - x_0) \in S^{n-1}$. Also, thanks to (16.2)-(16.3), properties (14.23)-(14.24) are satisfied. Since $u(x_0) = M < u(x)$ for each $x \in \Omega \setminus U$, the conclusion in Theorem 14.3 applies with Ω replaced by $\Omega \setminus U$ and, say, $\vec{\ell} := h \in S^{n-1}$. In the current context, this yields

$$0 < (D^{(\inf)}_{\vec{\ell}} u)(x_0) = \vec{\ell} \cdot (\nabla u)(x_0), \tag{16.7}$$

which contradicts the first condition in (16.6). □

Remark 16.2. *In the original formulation of the SMP in Hopf's 1927 paper [42], the coefficient matrix of the top-order part of the differential operator L is assumed to be locally uniformly positive definite in Ω, and the drift coefficients locally bounded in Ω. See also [63, pp. 14-15], [82, p. 14]. The version of the SMP given in [79, Theorem 5 on p. 61 and Remark (i) on p. 64] and [30, p. 35] is slightly more general (and natural), in the sense that the conditions on the coefficients of the second and first order terms of L are*

$$(A(x)\xi) \cdot \xi > 0 \ \textit{ for each } \ x \in \Omega \ \textit{ and } \ \xi \in S^{n-1}, \quad \textit{and the quantities} \tag{16.8}$$

$$\frac{\operatorname{Tr} A(x)}{\min\limits_{\xi \in S^{n-1}} (A(x)\xi) \cdot \xi} \quad \textit{and} \quad \frac{|\vec{b}(x)|}{\min\limits_{\xi \in S^{n-1}} (A(x)\xi) \cdot \xi} \quad \textit{are locally bounded in } \Omega. \tag{16.9}$$

Compared with the status-quo, our main contribution in Theorem 16.1 is weakening (16.9) *to the blow-up condition for the coefficients formulated in* (16.3). *Of course, the key factor in this regard,* *is the more flexible version of the Boundary Point Principle proved in Theorem 14.3.*

Remark 16.3. *Theorem 16.1 readily implies a weak minimum principle of the following form. Let Ω be a nonempty, bounded, open subset of $\mathbb{R}^n$ and retain the same assumptions on L as in the statement of Theorem 16.1. Then, if $u \in \mathscr{C}^0(\overline{\Omega}) \cap \mathscr{C}^2(\Omega)$ is a function which satisfies the differential inequality $(Lu)(x) \geq 0$ for all $x \in \Omega$, one has*

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u. \tag{16.10}$$

Theorem 16.1 is sharp, in the sense which we now describe. Fix two numbers $\alpha > 1$, $\beta > 0$ and, for each $i \in \{1, ..., n\}$, define the function $b^i : B(0,1) \to \mathbb{R}$ by setting

$$b^i(x) := \begin{cases} (n+\beta)\dfrac{x_i}{|x|^\alpha} & \text{if } \ x \in B(0,1) \setminus \{0\}, \\ 0 \ \text{ if } \ x = 0. \end{cases} \tag{16.11}$$

Next, consider the differential operator

$$L := -\Delta + \sum_{i=1}^{n} b^i(x)\partial_i \qquad \text{in } \ B(0,1), \tag{16.12}$$

and note that if

$$u : \overline{B(0,1)} \to \mathbb{R}, \qquad u(x) := |x|^{2+\beta}, \quad \forall\, x \in \overline{B(0,1)}, \tag{16.13}$$

then

$$\begin{aligned} &u \in \mathscr{C}^2\big(\overline{B(0,1)}\big), \quad \nabla u(x) = (\beta+2)|x|^\beta x \quad \text{and} \\ &\Delta u(x) = (\beta+2)(n+\beta)|x|^\beta \ \text{ for each } \ x \in \overline{B(0,1)}. \end{aligned} \tag{16.14}$$

Moreover, u is a nonconstant function which attains its global minimum at the origin. More precisely,

$$u \geq 0 \ \text{ in } \ B(0,1), \quad u(0) = 0 \ \text{ and } \ u\big|_{\partial B(0,1)} = 1. \tag{16.15}$$

Furthermore,

$$(Lu)(0) = 0 \ \text{ and for each } \ x \in B(0,1) \setminus \{0\}, \tag{16.16}$$

$$(Lu)(x) = (\beta+2)(n+\beta)|x|^\beta\big[1 - |x|^{2-\alpha}\big], \tag{16.17}$$

which shows that

$$\alpha \geq 2 \Longleftrightarrow (Lu)(x) \geq 0 \;\text{ for each }\; x \in B(0,1). \tag{16.18}$$

On the other hand, given a function $\widetilde{\omega} : (0,1) \to (0,+\infty)$ and a vector $\xi \in S^{n-1}$, condition (16.3) entails

$$\limsup_{\substack{x\cdot\xi>0\\ x\to 0}} \frac{|x|^{-\alpha}\, x\cdot\xi}{\frac{\widetilde{\omega}(x\cdot\xi)}{x\cdot\xi}} < +\infty \tag{16.19}$$

which, when specialized to the case when x approaches 0 along the ray $\{t\xi : t > 0\}$, implies the existence of some constant $c \in (0,+\infty)$ such that $\widetilde{\omega}(t) \geq c\,t^{2-\alpha}$ for all small $t > 0$. In turn, this readily shows that

$$\exists\,\widetilde{\omega} : (0,1) \to (0,+\infty) \text{ such that (16.3) holds and } \int_0^1 \frac{\widetilde{\omega}(t)}{t}\,dt < +\infty \Longleftrightarrow \alpha < 2. \tag{16.20}$$

The bottom line is that, in the context of the situation considered above, the range of α's for which the conclusion in Theorem 16.1 fails is precisely the complement of the range of α's for which the blow-up condition described in (16.3) is violated (compare (16.18) with (16.20)). Hence, Theorem 16.1 is optimal.

Chapter 17

Applications to Boundary Value Problems

Of course, a direct corollary of the Strong Maximum Principle established in Chapter 16 is the uniqueness in the Dirichlet problem formulated in the geometrical-analytical context considered in Theorem 16.1. We aim at proving similar results for Neumann and oblique type boundary value problems.

In the subsequent discussion, suppose that Ω is a nonempty, open, proper subset of $\mathbb{R}^n$ which is of locally finite perimeter. Denote by $\partial^*\Omega$ the reduced boundary of Ω, and by $\nu : \partial^*\Omega \to S^{n-1}$ the geometric measure theoretic outward unit normal to Ω (cf. Chapter 10). In addition, consider a second-order, elliptic, differential operator L, in non-divergence form, as in (14.20). In this context, the goal is to assign a concrete meaning to the *conormal derivative associated with the operator* L, which is originally formally expressed (at boundary points) as

$$\partial_\nu^L := -\sum_{i,j=1}^n a^{ij}\nu_i\partial_j = \Big(-\sum_{i,j=1}^n a^{ij}\nu_i\,\mathbf{e}_j\Big)\cdot\nabla \tag{17.1}$$

where $(\nu_i)_{1\leq i\leq n}$ are the components of ν. To this end, fix a point $x_0 \in \partial^*\Omega$ and assume that

$$\begin{array}{l} L \text{ is uniformly elliptic near } x_0 \text{ and its top-order} \\ \text{coefficients may be continuously extended at } x_0. \end{array} \tag{17.2}$$

In this setting, define the vector

$$\mathbf{n} := \mathbf{n}(L,\Omega,x_0) := -\sum_{i,j=1}^{n} a^{ij}(x_0)\nu_i(x_0)\,\mathbf{e}_j \in \mathbb{R}^n \tag{17.3}$$

and note that, since $\nu(x_0)\in S^{n-1}$, we have

$$\mathbf{n}\cdot\nu(x_0) = -\sum_{i,j=1}^{n} a^{ij}(x_0)\nu_i(x_0)\nu_j(x_0) < 0. \tag{17.4}$$

In particular, this shows that $\mathbf{n}\neq 0$. Finally, make the assumption that, in the sense of Definition 14.1,

$$\mathbf{n} \text{ points in } \Omega \text{ at } x_0. \tag{17.5}$$

Then, given a function $u\in\mathscr{C}^0(\Omega\cup\{x_0\})\cap\mathscr{C}^1(\Omega)$, formula (17.3) and the second equality in (17.1) suggest defining

$$\partial_\nu^L u(x_0) := \big(D_{\mathbf{n}}^{(\inf)}u\big)(x_0). \tag{17.6}$$

Let us also agree to drop the dependence on L when writing ∂_ν^L in the special case when $L=-\Delta$, in which scenario $\partial_\nu := -\sum\limits_{i=1}^{n}\nu_i\partial_i$ will be simply referred to as the *inner normal derivative to* $\partial\Omega$.

Before concluding this preliminary discussion, we wish to note that

$$\begin{array}{c}\text{if } \Omega \text{ is of locally finite perimeter, satisfying an interior pseudo-ball}\\ \text{condition at } x_0\in\partial^*\Omega, \text{ and if } L \text{ is as in (17.2) then (17.5) holds.}\end{array} \tag{17.7}$$

Indeed, in this scenario Proposition 9.1 shows that $-\nu(x_0)\in S^{n-1}$ is the direction vector for the pseudo-ball at x_0. Then (17.5) follows from this and (17.4), by Theorem 14.3.

Proposition 17.1. *Suppose Ω is a nonempty, open, proper subset of $\mathbb{R}^n$ which is of locally finite perimeter. Denote by $\partial^*\Omega$ the reduced boundary of Ω, and by $\nu:\partial^*\Omega\to S^{n-1}$ the geometric measure theoretic outward unit normal to Ω. Assume that $x_0\in\partial^*\Omega$ is a point with the property that Ω satisfies an interior pseudo-ball condition at x_0 for a shape function $\omega:[0,R]\to[0,+\infty)$ satisfying* *the properties listed in* (14.18)-(14.19) *as well as Dini's integrability condition. Also, suppose that* $\vec{\ell} \in S^{n-1}$ *is a vector which is inner transversal to* $\partial\Omega$ *at* x_0*, in the sense that*

$$\vec{\ell} \cdot \nu(x_0) < 0. \tag{17.8}$$

Next, consider a second-order, differential operator L*, in non-divergence form, as in* (14.20)*, which is uniformly elliptic near* x_0 *and whose top-order coefficients, originally defined in* Ω*, may be continuously extended at the point* $x_0 \in \partial\Omega$*. In addition, assume that there exists a real-valued function* $\widetilde{\omega} \in \mathscr{C}^0([0,R])$*, positive on* $(0,R]$*, satisfying* $\int_0^R \frac{\widetilde{\omega}(t)}{t}\,dt < +\infty$*, and with the property that*

$$\limsup_{\substack{\Omega \ni x \to x_0 \\ (x-x_0)\cdot\nu(x_0)>0}} \frac{\max\Big\{0\,,\vec{b}(x)\cdot\nu(x_0)\Big\} + \Big(\sum\limits_{i=1}^{n} \max\big\{0\,,\,-b^i(x)\big\}\Big)\omega(|x-x_0|)}{\frac{\widetilde{\omega}((x-x_0)\cdot\nu(x_0))}{(x-x_0)\cdot\nu(x_0)}} < +\infty. \tag{17.9}$$

Finally, suppose that $u \in \mathscr{C}^0(\Omega \cup \{x_0\}) \cap \mathscr{C}^2(\Omega)$ *is a real-valued subsolution of* L *in* Ω *which has a strict global minimum at* x_0 *(in the sense of* (14.26)-(14.27)*). Then the vector* $\vec{\ell}$ *points inside* Ω *at* x_0 *and*

$$\big(D_{\vec{\ell}}^{(inf)} u\big)(x_0) > 0. \tag{17.10}$$

In particular, with ∂_ν *and* ∂_ν^L *denoting, respectively, the inner normal derivative to* $\partial\Omega$*, and the conormal derivative associated with* L*, one has*

$$(\partial_\nu u)(x_0) > 0 \quad \text{ and } \quad (\partial_\nu^L u)(x_0) > 0. \tag{17.11}$$

Proof. Proposition 9.1 shows that $-\nu(x_0) \in S^{n-1}$ is the direction vector for the pseudo-ball at x_0. Granted this, the inequality in (17.10) becomes a consequence of (14.30). Then the two inequalities in (17.11) are obtained by specializing (17.10), respectively, to the case when $\vec{\ell} := -\nu(x_0) \in S^{n-1}$, and to the case when

$$\vec{\ell} := -\frac{\sum_{i,j=1}^n a^{ij}(x_0)\nu_i(x_0)\,\mathbf{e}_j}{\Big|\sum_{i,j=1}^n a^{ij}(x_0)\nu_i(x_0)\,\mathbf{e}_j\Big|} \in S^{n-1}, \tag{17.12}$$

which is a well-defined unit vector satisfying (17.8) (by the uniform ellipticity of L). □

Corollary 17.2. *With the same background assumptions on the operator L and the function u as in Proposition 17.1, all earlier conclusions hold in domains of class $\mathscr{C}^{1,\omega}$ provided ω satisfies* (8.11), (14.18)-(14.19)*, as well as Dini's integrability condition.*

This is sharp, in the sense that there exists a bounded domain of class $\mathscr{C}^1$ (which is even convex and of class $\mathscr{C}^\infty$ near all but one of its boundary points) for which the aforementioned conclusions fail.

Proof. The claim in the first part of the statement is a direct consequence of Theorem 12.3 and Proposition 17.1. Its sharpness is implied by the counterexamples described earlier, in Remark 15.9 and Remark 15.8. □

Theorem 17.3. *Suppose that $\Omega \subseteq \mathbb{R}^n$ is a nonempty, open, connected, bounded set and consider a second-order, elliptic differential operator L, in non-divergence form in Ω, as in* (14.20)*. Also, suppose that there exists a family of real-valued functions $\widetilde{\omega}_{x,\xi} \in \mathscr{C}^0\big([0,1]\big)$, indexed by $x \in \overline{\Omega}$ and $\xi \in S^{n-1}$, each positive on $(0,1)$ and satisfying Dini's integrability condition, such that the following two properties hold:*

(i) for each $x \in \partial\Omega$ there exists $h = h_x \in S^{n-1}$ so that Ω satisfies an interior pseudo-ball condition at x with shape function $\omega = \omega_x$ satisfying the properties listed in (14.18)-(14.19)*, and direction vector h, for which*

$$\limsup_{\substack{\Omega\ni y\to x\\ (y-x)\cdot h>0}} \frac{\frac{\omega(|y-x|)}{|y-x|}\Big(\mathrm{Tr}\,A(y)\Big) + \max\Big\{0\,,\vec{b}(y)\cdot h\Big\} + \Big(\sum_{i=1}^{n}\max\big\{0\,,\,-b^i(y)\big\}\Big)\omega(|y-x|)}{\frac{\widetilde{\omega}_{x,h}((y-x)\cdot h)}{(y-x)\cdot h}\Big((A(y)h)\cdot h\Big)}, \qquad (17.13)$$

is finite;

(ii) for each $x \in \Omega$ and each $\xi \in S^{n-1}$, there holds

$$\limsup_{\Omega\ni y\to x} \frac{\Big(\mathrm{Tr}\,A(y)\Big) + \max\Big\{0\,,\vec{b}(y)\cdot\xi\Big\} + \Big(\sum_{i=1}^{n}\max\big\{0\,,\,-b^i(y)\big\}\Big)|y-x|}{\frac{\widetilde{\omega}_{x,\xi}((y-x)\cdot\xi)}{(y-x)\cdot\xi}\Big((A(y)\xi)\cdot\xi\Big)}, \qquad (17.14)$$

is finite.

Finally, assume that $\vec{\ell}: \partial\Omega \to S^{n-1}$ *is a vector field with the property that*

$$\vec{\ell}(x) \cdot h_x > 0 \quad \textit{for each} \;\; x \in \partial\Omega. \tag{17.15}$$

Then for each $u \in \mathscr{C}^0(\overline{\Omega}) \cap \mathscr{C}^2(\Omega)$ *one has*

$$u \text{ is constant in } \overline{\Omega} \iff \begin{cases} (Lu)(x) \geq 0 \quad \textit{for each} \;\; x \in \Omega, \\ \big(D^{(inf)}_{\vec{\ell}(x)} u\big)(x) \leq 0 \quad \textit{for each} \;\; x \in \partial\Omega. \end{cases} \tag{17.16}$$

In particular, one has uniqueness for the oblique derivative boundary value problem for L *in* Ω*, i.e., for any given data* $f : \Omega \to \mathbb{R}$*,* $g : \partial\Omega \to \mathbb{R}$*, there is at most one function* u *satisfying*

$$\begin{cases} u \in \mathscr{C}^1(\overline{\Omega}) \cap \mathscr{C}^2(\Omega), \\ (Lu)(x) = f(x) \quad \textit{for each} \;\; x \in \Omega, \\ \vec{\ell}(x) \cdot (\nabla u)(x) = g(x) \;\; \textit{for each } x \in \partial\Omega. \end{cases} \tag{17.17}$$

As a consequence, if Ω *is also of finite perimeter and has the property that* $\partial^*\Omega = \partial\Omega$*, and if* L *is actually uniformly elliptic and its top-order coefficients belong to* $\mathscr{C}^0(\overline{\Omega})$*, then*

$$u \in \mathscr{C}^1(\overline{\Omega}) \cap \mathscr{C}^2(\Omega), \quad Lu \geq 0 \;\; \textit{in} \;\; \Omega, \;\; \textit{and} \tag{17.18}$$

$$\partial^L_\nu u \leq 0 \;\; \textit{on} \;\; \partial\Omega \implies u \text{ is constant in } \overline{\Omega}. \tag{17.19}$$

Hence, in this setting, one has uniqueness for the Neumann boundary value problem for L *in* Ω*, i.e., for any given data* f, g *there is at most one function* u *satisfying*

$$\begin{cases} u \in \mathscr{C}^1(\overline{\Omega}) \cap \mathscr{C}^2(\Omega), \\ Lu = f \quad \textit{in} \;\; \Omega, \\ \partial^L_\nu u = g \;\; \textit{on} \;\; \partial\Omega. \end{cases} \tag{17.20}$$

Finally, all these results are sharp in the sense that, even in the class of uniformly elliptic operators with constant top coefficients, condition (17.14) *may not be relaxed to*

$$\limsup_{\Omega \ni y \to x} \big[|x - y||\vec{b}(y)|\big] < +\infty, \qquad \forall\, x \in \overline{\Omega}. \tag{17.21}$$

Proof. As a preliminary matter, we note that (17.15) and the fact that, by (i), Ω satisfies an interior pseudo-ball condition at each $x \in \partial\Omega$ with direction vector $h_x \in S^{n-1}$, imply that $\vec{\ell}(x)$ points inside

Ω for each $x \in \partial\Omega$ (cf. the proof of Theorem 14.3). In particular, $\big(D^{(\inf)}_{\vec{\ell}(x)} u\big)(x)$ is well-defined for each $x \in \partial\Omega$. To proceed, assume that $u \in \mathscr{C}^0(\overline{\Omega}) \cap \mathscr{C}^2(\Omega)$ attains a strict global minimum on $\partial\Omega$, i.e., there exists a point $x_0 \in \partial\Omega$ such that $u(x_0) < u(x)$ for all $x \in \Omega$. In this case, granted property (i) in the statement of the theorem, Theorem 14.5 yields $\big(D^{(\inf)}_{\vec{\ell}(x_0)} u\big)(x_0) > 0$, contradicting the second condition in the right-hand side of (17.16). Thus, $u \in \mathscr{C}^0(\overline{\Omega})$ attains its minimum in Ω. In concert with the assumption that Ω is connected, property (ii) in the statement of the theorem, and the fact that $Lu \geq 0$ in Ω, the Strong Maximum Principle established in Theorem 16.1 allows us to conclude u is constant in Ω. This proves (17.16) which, in turn, readily yields uniqueness in the oblique boundary value problem (17.17).

As far as (17.18) is concerned, the fact that $\partial^*\Omega = \partial\Omega$ ensures that the geometric measure theoretic outward unit normal ν to Ω is everywhere defined on $\partial\Omega$. Thus, if the top-order coefficients of L belong to $\mathscr{C}^0(\overline{\Omega})$, we may define

$$\vec{\ell} : \partial\Omega \longrightarrow S^{n-1}, \quad \vec{\ell}(x) := -\frac{\sum_{i,j=1}^n a^{ij}(x)\nu_i(x)\,\mathbf{e}_j}{\Big|\sum_{i,j=1}^n a^{ij}(x)\nu_i(x)\,\mathbf{e}_j\Big|} \quad \text{for every} \quad x \in \partial\Omega. \tag{17.22}$$

Now (17.18) follows by specializing (17.16) to this choice of a vector field.

Finally, to see that the above results are sharp, take $\Omega := B(0,1) \subseteq \mathbb{R}^n$ and consider the differential operator L and the function $u \in \mathscr{C}^2\big(\overline{B(0,1)}\big)$ as in (1.41). Then

$$(Lu)(x) = 0 \ \text{ for each } \ x \in B(0,1), \quad \text{and} \tag{17.23}$$

$$(\partial^L_\nu u)(x) = -\tfrac{4}{n+2} \leq 0 \ \text{ for each } \ x \in \partial B(0,1) \tag{17.24}$$

which shows that (17.18) fails in this case, precisely because the blow-up of the drift at the origin is of order one, i.e., $|\vec{b}(x)| = |x|^{-1}$ for $x \in B(0,1) \setminus \{0\}$. $\square$

Corollary 17.4. *With the same background assumptions on the operator L and the function u as* *in Theorem 17.3, all conclusions in this theorem hold in bounded connected domains of class* $\mathscr{C}^{1,\omega}$ *in* $\mathbb{R}^n$ *provided* ω *satisfies* (8.11), (14.18)-(14.19), *as well as Dini's integrability condition.*

Proof. This readily follows from Theorem 12.3 and Theorem 17.3. $\square$

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