STUDIA MATHEMATICA
Online First version

# Pointwise characterization of Besov and Triebel–Lizorkin spaces on spaces of homogeneous type

by

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**Abstract.** The authors establish a pointwise characterization of Besov and Triebel–Lizorkin spaces on spaces of homogeneous type via clarifying the relation between Hajłasz–Sobolev spaces, Hajłasz–Besov and Hajłasz–Triebel–Lizorkin spaces, grand Besov and Triebel–Lizorkin spaces, and Besov and Triebel–Lizorkin spaces. A major novelty of this article is that all results presented get rid of both the dependence on the reverse doubling condition of the measure and the metric condition of the quasi-metric under consideration. Moreover, the pointwise characterization of the inhomogeneous version is new even when the underlying space is an RD-space.

**1. Introduction.** It is well known that Besov and Triebel–Lizorkin spaces provide a unified frame for the study of many function spaces and indeed cover many well-known classical concrete function spaces such as Lebesgue spaces, Sobolev spaces, potential spaces, (local) Hardy spaces, and the space of functions with bounded mean oscillation. We refer the reader to the monographs [5, 44, 45, 46, 47] for a comprehensive treatment of these function spaces and their history. We also refer the reader to [61] for relations among Morrey spaces, Campanato spaces, and Besov–Triebel–Lizorkin spaces, to [3, 40] for some new progress of Besov and Triebel–Lizorkin spaces, and to [6, 7, 8, 9] for various characterizations and applications of Besov and Triebel–Lizorkin spaces associated with operators.

In particular, fractional Sobolev spaces play an important major role in many questions involving partial differential equations on $\mathbb{R}^n$. It is known that a theory of first order Sobolev spaces on doubling metric spaces has been established based on both upper gradients [31, 41] and pointwise in-

2020 *Mathematics Subject Classification*: Primary 46E36; Secondary 46E35, 42B25, 30L99.
*Key words and phrases*: space of homogeneous type, Besov space, Triebel–Lizorkin space, Hajłasz–Sobolev space, Hajłasz–Triebel–Lizorkin space, grand Triebel–Lizorkin space, pointwise characterization.
Received 21 June 2021; revised 22 April 2022.
Published online *.

 

equalities [14]; see [15, 16] for a survey on this. These different approaches result in the same function class if the underlying space supports a suitable Poincaré inequality [35]. In the present article, we further investigate the spaces introduced by Hajłasz [14] (see also [55] and Definition 2.10 below), which are defined via pointwise inequalities.

On the other hand, as a generalization of $\mathbb{R}^n$, spaces of homogeneous type were introduced by Coifman and Weiss [10, 11] (see Definition 2.2 below), which provides a natural setting for the study of function spaces and the boundedness of Calderón–Zygmund operators. We refer the reader to [42, 43, 50, 51] for some very recent progress on function spaces on spaces of homogeneous type. Function spaces and their applications on spaces of homogeneous type, with some additional assumptions, have been extensively investigated in many articles. For instance, the *Ahlfors d-regular space* is a space of homogeneous type satisfying the following condition: there exists a positive constant $C$ such that, for any ball $B(x,r) \subset \mathcal{X}$ with center $x$ and radius $r \in (0, \operatorname{diam} \mathcal{X})$,

$$C^{-1} r^d \le \mu(B(x,r)) \le C r^d;$$

here and hereafter, $\operatorname{diam} \mathcal{X} := \sup_{x,y \in \mathcal{X}} d(x,y)$. Another case is the *RD-space* (see [19, 20, 34] for instance), which is a doubling metric measure space satisfying the following additional *reverse doubling condition*: there exist positive constants $\widetilde{C}_{(\mu)} \in (0,1]$ and $\kappa \in (0, \omega]$ such that, for any ball $B(x,r)$ with $r \in (0, \operatorname{diam} \mathcal{X}/2)$ and $\lambda \in [1, \operatorname{diam} \mathcal{X}/(2r))$,

$$\widetilde{C}_{(\mu)} \lambda^\kappa \mu(B(x,r)) \le \mu(B(x, \lambda r)).$$

Obviously, an RD-space is a generalization of an Ahlfors $d$-regular space. We refer the reader to [60] for more equivalent characterizations of RD-spaces.

Besov and Triebel–Lizorkin spaces on spaces of homogeneous type satisfying some additional assumptions were also studied. We refer the reader to [21, 22, 56, 58, 59] for various characterizations of Besov and Triebel–Lizorkin spaces on Ahlfors $d$-regular spaces, and to [52, 53, 54, 57] for some applications. We also refer the reader to [20, 38, 60] for various characterizations of Besov and Triebel–Lizorkin spaces on RD-spaces. Moreover, Koskela et al. [36, 37] introduced Hajłasz–Besov and Hajłasz–Triebel–Lizorkin spaces on RD-spaces. We refer the reader to [12, 27, 28, 29, 30] for various characterizations and applications of Hajłasz–Besov and Hajłasz–Triebel–Lizorkin spaces on a metric measure space satisfying the doubling property.

Recently, using the wavelet reproducing formulae of [18], Han et al. [17] introduced Besov and Triebel–Lizorkin spaces on spaces of homogeneous type and established several embedding theorems. On the other hand, Wang et al. [48] also introduced Besov and Triebel–Lizorkin spaces on spaces of homogeneous type, based on the Calderón reproducing formulae established in [24], and established the boundedness of Calderón–Zygmund operators on these spaces as an application. Later, He et al. [25] obtained characterizations of Besov and Triebel–Lizorkin spaces via wavelets, molecules, Lusin area functions, and Littlewood–Paley $g_\lambda^*$-functions. Moreover, He et al. [25] showed that those two kinds of Besov and Triebel–Lizorkin spaces studied, respectively, in [17] and [48] coincide. Then Wang et al. [49] established the difference characterization of Besov and Triebel–Lizorkin spaces on spaces of homogeneous type.

To complete the theory of Besov and Triebel–Lizorkin spaces on spaces of homogeneous type, it is a natural question whether or not we can also establish a pointwise characterization of Besov and Triebel–Lizorkin spaces on spaces of homogeneous type. The main target of this article is to give an affirmative answer to this question.

The organization of this article is as follows.

In Section 2, we first recall the concepts of both homogeneous Besov and Triebel–Lizorkin spaces on spaces of homogeneous type introduced in [48], and then introduce Hajłasz–Sobolev spaces and Hajłasz–Besov–Triebel–Lizorkin spaces on spaces of homogeneous type. Then we state the main result of this article (Theorem 2.16).

In Section 3, we first introduce homogeneous grand Besov and Triebel–Lizorkin spaces on spaces of homogeneous type. Then we investigate the relation between homogeneous grand Besov and Triebel–Lizorkin spaces and homogeneous Besov and Triebel–Lizorkin spaces (Theorem 3.3). Later, we establish the equivalence between homogeneous Hajłasz–Besov–Triebel–Lizorkin spaces and homogeneous grand Besov and Triebel–Lizorkin spaces (Theorem 3.10). To this end, we first establish a Poincaré type inequality (Lemma 3.11). It should be mentioned that, in the proof of Lemma 3.11, the constant $A_0$ appearing in the quasi-triangle inequality (see Definition 2.1) also brings some difficulty. That is why we need additional restrictions on parameters involved therein. Moreover, all the proofs in Section 3 get rid of the dependence on the reverse doubling assumption.

In Section 4, we establish the equivalence between inhomogeneous Hajłasz–Besov–Triebel–Lizorkin spaces and inhomogeneous Besov–Triebel–Lizorkin spaces (Theorem 4.10). To this end, we first establish a new characterization of inhomogeneous Besov–Triebel–Lizorkin spaces (Theorem 4.11).

Finally, let us make some conventions on notation. For any given $p \in (0,\infty]$, the *Lebesgue space* $L^p(\mathcal{X})$ is defined by setting

$$L^p(\mathcal{X}) := \{f \text{ measurable on } \mathcal{X} : \|f\|_{L^p(\mathcal{X})} < \infty\},$$

where, for any measurable function $f$ on $\mathcal{X}$,

$$\|f\|_{L^p(\mathcal{X})} := \begin{cases} \left[\int_{\mathcal{X}} |f(x)|^p \, d\mu(x)\right]^{1/p} & \text{if } p \in (0,\infty), \\ \operatorname*{ess\,sup}_{x\in\mathcal{X}} |f(x)| & \text{if } p = \infty. \end{cases}$$

Throughout this article, we use $A_0$ to denote the positive constant appearing in the *quasi-triangle inequality* of $d$ (see Definition 2.1), the parameter $\omega$ to denote the *upper dimension* in Definition 2.2 [see (2.2)], and $\eta$ to denote the smoothness index of the exp-ATI in Definition 2.5. Moreover, $\delta$ is a small positive number, for instance, $\delta \le (2A_0)^{-10}$, coming from the construction of the dyadic cubes on $\mathcal{X}$ (see Lemma 2.4). For any given $p \in [1,\infty]$, we use $p'$ to denote its conjugate index, that is, $1/p + 1/p' = 1$. For any $r \in \mathbb{R}$, $r_+$ is defined by setting $r_+ := \max\{0, r\}$. For any $a, b \in \mathbb{R}$, let $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. The symbol $C$ denotes a positive constant which is independent of the main parameters involved, but may vary from line to line. We use $C_{(\alpha,\beta,\dots)}$ to denote a positive constant depending on the indicated parameters $\alpha, \beta, \dots$. The symbol $A \lesssim B$ means that $A \le CB$ for some positive constant $C$, while $A \sim B$ means $A \lesssim B \lesssim A$. If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, *rather than* $f \lesssim g = h$ or $f \lesssim g \le h$. We let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$. For any $r \in (0,\infty)$ and $x, y \in \mathcal{X}$ with $x \ne y$, define $V(x, y) := \mu(B(x, d(x, y)))$ and $V_r(x) := \mu(B(x, r))$. For any $\beta, \gamma \in (0, \eta)$ and $s \in (-(\beta \wedge \gamma), \beta \wedge \gamma)$, we always let

$$p(s, \beta \wedge \gamma) := \max\left\{\frac{\omega}{\omega + (\beta \wedge \gamma)}, \frac{\omega}{\omega + (\beta \wedge \gamma) + s}\right\}, \tag{1.1}$$

where $\omega$ and $\eta$ are, respectively, as in (2.2) and Definition 2.5. The operator $M$ always denotes the *central Hardy–Littlewood maximal operator*, which is defined by setting, for any locally integrable function $f$ on $\mathcal{X}$ and any $x \in \mathcal{X}$,

$$M(f)(x) := \sup_{r\in(0,\infty)} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |f(y)| \, d\mu(y). \tag{1.2}$$

For any set $E \subset \mathcal{X}$, we use $\mathbf{1}_E$ to denote its characteristic function, and for any set $J$, we use $\#J$ to denote its cardinality.

**2. Besov and Triebel–Lizorkin spaces and Hajłasz–Sobolev spaces on spaces of homogeneous type.** In this section, we recall the concepts of Besov and Triebel–Lizorkin spaces on spaces of homogeneous type and introduce Hajłasz–Sobolev spaces on spaces of homogeneous type. Let us begin with the concept of quasi-metric spaces.

Definition 2.1. A *quasi-metric space* $(\mathcal{X}, d)$ is a non-empty set $\mathcal{X}$ equipped with a *quasi-metric* $d$, that is, a non-negative function defined on $\mathcal{X} \times \mathcal{X}$ satisfying, for any $x, y, z \in \mathcal{X}$,

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) there exists a constant $A_0 \in [1, \infty)$, independent of $x, y, z$, such that

$$d(x, z) \le A_0[d(x, y) + d(y, z)].$$

The *ball* $B$ of $\mathcal{X}$, centered at $x_0 \in \mathcal{X}$ with radius $r \in (0, \infty)$, is defined by setting

$$B := B(x_0, r) := \{x \in \mathcal{X} : d(x, x_0) < r\}.$$

For any ball $B$ and $\tau \in (0, \infty)$, we denote by $\tau B$ the ball with the same center as that of $B$ but of radius $\tau$ times that of $B$.

Definition 2.2. Let $(\mathcal{X}, d)$ be a quasi-metric space and $\mu$ a non-negative measure on $\mathcal{X}$. The triple $(\mathcal{X}, d, \mu)$ is called a *space of homogeneous type* if $\mu$ satisfies the following doubling condition: there exists a positive constant $C \in [1, \infty)$ such that, for any ball $B \subset \mathcal{X}$,

$$0 < \mu(2B) \le C\mu(B) < \infty. \tag{2.1}$$

Let $C_{(\mu)} := \sup_{B\subset\mathcal{X}} \mu(2B)/\mu(B)$. Then it is easy to show that $C_{(\mu)}$ is the smallest positive constant such that (2.1) holds true. The above doubling condition implies that, for any ball $B$ and any $\lambda \in [1, \infty)$,

$$\mu(\lambda B) \le C_{(\mu)}\lambda^{\omega}\mu(B), \tag{2.2}$$

where $\omega := \log_2 C_{(\mu)}$ is called the *upper dimension* of $\mathcal{X}$. Note that $\omega \in (0, \infty)$ (see, for instance, [2, p. 72]). If $A_0 = 1$, then $(\mathcal{X}, d, \mu)$ is called a *metric measure space of homogeneous type*, or simply a *doubling metric measure space.*

Without loss of generality, we may make the following assumptions on $(\mathcal{X}, d, \mu)$. For any point $x \in \mathcal{X}$, we assume that the balls $\{B(x, r)\}_{r\in(0,\infty)}$ form a basis of open neighborhoods of $x$. Moreover, we suppose that $\mu$ is *Borel regular*, which means that open sets are measurable and every set $A \subset \mathcal{X}$ is contained in a Borel set $E$ satisfying $\mu(A) = \mu(E)$. We also assume $\mu(B(x, r)) \in (0, \infty)$ and $\mu(\{x\}) = 0$ for any given $x \in \mathcal{X}$ and $r \in (0, \infty)$.

Now, we recall the concepts of both test functions and distributions on $\mathcal{X}$, originally introduced in [20] (see also [19]).

Definition 2.3 (Test functions). Let $x_1 \in \mathcal{X}$, $r \in (0, \infty)$, $\beta \in (0, 1]$, and $\gamma \in (0, \infty)$. For any $x \in \mathcal{X}$, define

$$D_\gamma(x_1, x; r) := \frac{1}{V_r(x_1) + V(x_1, x)}\left[\frac{r}{r + d(x_1, x)}\right]^\gamma. \tag{2.3}$$

A measurable function $f$ on $\mathcal{X}$ is called a *test function of type* $(x_1, r, \beta, \gamma)$ if there exists a positive constant $C$ such that

(i) for any $x \in \mathcal{X}$,

$$|f(x)| \leq CD_\gamma(x_1, x; r); \tag{2.4}$$

(ii) for any $x, y \in \mathcal{X}$ satisfying $d(x,y) \leq (2A_0)^{-1}[r + d(x_1, x)]$,

$$|f(x) - f(y)| \leq C\left[\frac{d(x,y)}{r + d(x_1,x)}\right]^\beta D_\gamma(x_1, x; r). \tag{2.5}$$

The set of all test functions of type $(x_1, r, \beta, \gamma)$ is denoted by $\mathcal{G}(x_1, r, \beta, \gamma)$. For any $f \in \mathcal{G}(x_1, r, \beta, \gamma)$, its norm $\|f\|_{\mathcal{G}(x_1,r,\beta,\gamma)}$ in $\mathcal{G}(x_1, r, \beta, \gamma)$ is defined by setting

$$\|f\|_{\mathcal{G}(x_1,r,\beta,\gamma)} := \inf\{C \in (0,\infty) : (2.4) \text{ and } (2.5) \text{ hold true}\}.$$

The subspace $\mathring{\mathcal{G}}(x_1, r, \beta, \gamma)$ is defined by setting

$$\mathring{\mathcal{G}}(x_1, r, \beta, \gamma) := \left\{f \in \mathcal{G}(x_1, r, \beta, \gamma) : \int_{\mathcal{X}} f(x)\, d\mu(x) = 0\right\}$$

and is equipped with the norm $\|\cdot\|_{\mathring{\mathcal{G}}(x_1,r,\beta,\gamma)} := \|\cdot\|_{\mathcal{G}(x_1,r,\beta,\gamma)}$.

Note that, for any fixed $x_1, x_2 \in \mathcal{X}$ and $r_1, r_2 \in (0,\infty)$, $\mathcal{G}(x_1, r_1, \beta, \gamma) = \mathcal{G}(x_2, r_2, \beta, \gamma)$ and $\mathring{\mathcal{G}}(x_1, r_1, \beta, \gamma) = \mathring{\mathcal{G}}(x_2, r_2, \beta, \gamma)$ with equivalent norms, but the positive equivalence constants may depend on $x_1$, $x_2$, $r_1$, and $r_2$. Thus, for fixed $x_0 \in \mathcal{X}$ and $r_0 = 1$, we may denote $\mathcal{G}(x_0, 1, \beta, \gamma)$ and $\mathring{\mathcal{G}}(x_0, 1, \beta, \gamma)$ simply by $\mathcal{G}(\beta, \gamma)$ and $\mathring{\mathcal{G}}(\beta, \gamma)$, respectively. Usually, the spaces $\mathcal{G}(\beta, \gamma)$ and $\mathring{\mathcal{G}}(\beta, \gamma)$ are called the *spaces of test functions* on $\mathcal{X}$.

Fix an $\varepsilon \in (0,1]$ and $\beta, \gamma \in (0, \varepsilon]$. Let $\mathcal{G}_0^\varepsilon(\beta, \gamma)$ [resp., $\mathring{\mathcal{G}}_0^\varepsilon(\beta, \gamma)$] be the completion of the set $\mathcal{G}(\varepsilon, \varepsilon)$ [resp., $\mathring{\mathcal{G}}(\varepsilon, \varepsilon)$] in $\mathcal{G}(\beta, \gamma)$ [resp., $\mathring{\mathcal{G}}(\beta, \gamma)$]. Furthermore, the norm of $\mathcal{G}_0^\varepsilon(\beta, \gamma)$ [resp., $\mathring{\mathcal{G}}_0^\varepsilon(\beta, \gamma)$] is defined by setting $\|\cdot\|_{\mathcal{G}_0^\varepsilon(\beta,\gamma)} := \|\cdot\|_{\mathcal{G}(\beta,\gamma)}$ [resp., $\|\cdot\|_{\mathring{\mathcal{G}}_0^\varepsilon(\beta,\gamma)} := \|\cdot\|_{\mathcal{G}(\beta,\gamma)}$]. The dual space $(\mathcal{G}_0^\varepsilon(\beta, \gamma))'$ [resp., $(\mathring{\mathcal{G}}_0^\varepsilon(\beta, \gamma))'$] is defined to be the set of all continuous linear functionals from $\mathcal{G}_0^\varepsilon(\beta, \gamma)$ [resp., $\mathring{\mathcal{G}}_0^\varepsilon(\beta, \gamma)$] to $\mathbb{C}$, equipped with the weak-$*$ topology. The spaces $(\mathcal{G}_0^\varepsilon(\beta, \gamma))'$ and $(\mathring{\mathcal{G}}_0^\varepsilon(\beta, \gamma))'$ are called the *spaces of distributions* on $\mathcal{X}$.

The following lemma, which comes from [32, Theorem 2.2], establishes the dyadic cube system of $(\mathcal{X}, d, \mu)$.

LEMMA 2.4. *Let constants $0 < c_0 \leq C_0 < \infty$ and $\delta \in (0,1)$ be such that $12A_0^3 C_0 \delta \leq c_0$. Assume that a set of points, $\{z_\alpha^k : k \in \mathbb{Z}, \alpha \in \mathcal{A}_k\} \subset \mathcal{X}$ with $\mathcal{A}_k$ for any $k \in \mathbb{Z}$ being a set of indices, has the following properties: for any $k \in \mathbb{Z}$,*

$$d(z_\alpha^k, z_\beta^k) \geq c_0\delta^k \quad \text{if } \alpha \neq \beta, \quad \text{and} \quad \min_{\alpha \in \mathcal{A}_k} d(x, z_\alpha^k) < C_0\delta^k \quad \text{for any } x \in \mathcal{X}.$$

*Then there exists a family of sets, $\{Q_\alpha^k : k \in \mathbb{Z},\ \alpha \in \mathcal{A}_k\}$, satisfying*

(i) *for any $k \in \mathbb{Z}$, $\bigcup_{\alpha\in\mathcal{A}_k} Q_\alpha^k = \mathcal{X}$ and $\{Q_\alpha^k : \alpha \in \mathcal{A}_k\}$ consists of mutually disjoint sets;*

(ii) *if $l, k \in \mathbb{Z}$ and $k \le l$, then, for any $\alpha \in \mathcal{A}_k$ and $\beta \in \mathcal{A}_l$, either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \emptyset$;*

(iii) *for any $k \in \mathbb{Z}$ and $\alpha \in \mathcal{A}_k$,*

$$B(z_\alpha^k, (3A_0^2)^{-1}c_0\delta^k) \subset Q_\alpha^k \subset B(z_\alpha^k, 2A_0C_0\delta^k).$$

Throughout this article, for any $k \in \mathbb{Z}$, define

$$\mathcal{G}_k := \mathcal{A}_{k+1} \setminus \mathcal{A}_k \quad \text{and} \quad \mathcal{Y}^k := \{z_\alpha^{k+1}\}_{\alpha\in\mathcal{G}_k} =: \{y_\alpha^k\}_{\alpha\in\mathcal{G}_k}$$

and, for any $x \in \mathcal{X}$, define

$$d(x, \mathcal{Y}^k) := \inf_{y\in\mathcal{Y}^k} d(x, y) \quad \text{and} \quad V_{\delta^k}(x) := \mu(B(x, \delta^k)).$$

Now, we recall from [24] the concept of approximations of the identity with exponential decay.

Definition 2.5. A sequence $\{Q_k\}_{k\in\mathbb{Z}}$ of bounded linear integral operators on $L^2(\mathcal{X})$ is called an *approximation of the identity with exponential decay* (for short, exp-ATI) if there exist constants $C$, $\nu \in (0, \infty)$, $a \in (0, 1]$, and $\eta \in (0, 1)$ such that, for any $k \in \mathbb{Z}$, the kernel of the operator $Q_k$, a function on $\mathcal{X} \times \mathcal{X}$, which is still denoted by $Q_k$, satisfies the following conditions:

(i) (the *identity condition*) $\sum_{k=-\infty}^{\infty} Q_k = I$ in $L^2(\mathcal{X})$, where $I$ denotes the *identity operator* on $L^2(\mathcal{X})$;

(ii) (the *size condition*) for any $x, y \in \mathcal{X}$,

$$|Q_k(x, y)| \le C\frac{1}{\sqrt{V_{\delta^k}(x)V_{\delta^k}(y)}}H_k(x, y),$$

where

$$H_k(x, y) := \exp\left\{-\nu\left[\frac{d(x, y)}{\delta^k}\right]^a\right\} \times \exp\left\{-\nu\left[\frac{\max\{d(x, \mathcal{Y}^k), d(y, \mathcal{Y}^k)\}}{\delta^k}\right]^a\right\};$$

(iii) (the *regularity condition*) for any $x, x', y \in \mathcal{X}$ with $d(x, x') \le \delta^k$,

$$|Q_k(x, y) - Q_k(x', y)| + |Q_k(y, x) - Q_k(y, x')| \le C\left[\frac{d(x, x')}{\delta^k}\right]^\eta \frac{1}{\sqrt{V_{\delta^k}(x)V_{\delta^k}(y)}}H_k(x, y);$$

(iv) (the *second difference regularity condition*) for any $x, x', y, y' \in \mathcal{X}$ with $d(x, x') \le \delta^k$ and $d(y, y') \le \delta^k$,

$$|[Q_k(x,y) - Q_k(x',y)] - [Q_k(x,y') - Q_k(x',y')]| \le C\left[\frac{d(x,x')}{\delta^k}\right]^\eta \left[\frac{d(y,y')}{\delta^k}\right]^\eta \frac{1}{\sqrt{V_{\delta^k}(x)V_{\delta^k}(y)}} H_k(x,y);$$

(v) (the *cancellation condition*) for any $x, y \in \mathcal{X}$,

$$\int_{\mathcal{X}} Q_k(x,y')\,d\mu(y') = 0 = \int_{\mathcal{X}} Q_k(x',y)\,d\mu(x').$$

The existence of such an exp-ATI on spaces of homogeneous type is guaranteed by [4, Theorem 7.1] with $\eta$ as in [4, Theorem 3.1], which might be very small (see also [24, Remark 2.8(i)]). However, if $d$ is a metric, then $\eta$ can be taken arbitrarily close to 1 (see [33, Corollary 6.13]).

The following lemma states some basic properties of exp-ATIs. One can find more details in [24, Remarks 2.8 and 2.9, and Proposition 2.10].

LEMMA 2.6. *Let* $\{Q_k\}_{k\in\mathbb{Z}}$ *be an* exp-ATI *and* $\eta \in (0,1)$ *be as in Definition* 2.5. *Then, for any given* $\Gamma \in (0,\infty)$, *there exists a positive constant* $C$ *such that, for any* $k \in \mathbb{Z}$, *the kernel* $Q_k$ *has the following properties:*

(i) *for any* $x, y \in \mathcal{X}$,

$$|Q_k(x,y)| \le C D_\Gamma(x,y;\delta^k), \tag{2.6}$$

*where* $D_\Gamma(x,y;\delta^k)$ *is as in* (2.3) *with* $\gamma$ *replaced by* $\Gamma$;

(ii) *for any* $x, x', y \in \mathcal{X}$ *with* $d(x,x') \le (2A_0)^{-1}[\delta^k + d(x,y)]$,

$$|Q_k(x,y) - Q_k(x',y)| + |Q_k(y,x) - Q_k(y,x')| \le C\left[\frac{d(x,x')}{\delta^k + d(x,y)}\right]^\eta D_\Gamma(x,y;\delta^k); \tag{2.7}$$

(iii) *for any* $x, x', y, y' \in \mathcal{X}$ *with* $d(x,x') \le (2A_0)^{-2}[\delta^k + d(x,y)]$ *and* $d(y,y') \le (2A_0)^{-2}[\delta^k + d(x,y)]$,

$$|[Q_k(x,y) - Q_k(x',y)] - [Q_k(x,y') - Q_k(x',y')]| \le C\left[\frac{d(x,x')}{\delta^k + d(x,y)}\right]^\eta \left[\frac{d(y,y')}{\delta^k + d(x,y)}\right]^\eta D_\Gamma(x,y;\delta^k).$$

Based on exp-ATIs, we now recall the concepts of Besov and Triebel–Lizorkin spaces on spaces of homogeneous type; see [48, Definitions 3.1 and 5.1].

DEFINITION 2.7. Let $\beta, \gamma \in (0,\eta)$ with $\eta$ as in Definition 2.5, and $s \in (-(\beta\wedge\gamma), \beta\wedge\gamma)$. Let $\{Q_k\}_{k\in\mathbb{Z}}$ be an exp-ATI.

(i) Let $p \in (p(s, \beta \wedge \gamma), \infty]$, with $p(s, \beta \wedge \gamma)$ as in (1.1), and $q \in (0, \infty]$. The *homogeneous Besov space* $\dot{B}^s_{p,q}(\mathcal{X})$ is defined by setting

$$\dot{B}^s_{p,q}(\mathcal{X}) := \{f \in (\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))' : \|f\|_{\dot{B}^s_{p,q}(\mathcal{X})} < \infty\},$$

where, for any $f \in (\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))'$,

$$\|f\|_{\dot{B}^s_{p,q}(\mathcal{X})} := \Big[\sum_{k=-\infty}^{\infty} \delta^{-ksq} \|Q_k(f)\|^q_{L^p(\mathcal{X})}\Big]^{1/q}$$

with the usual modifications when $q = \infty$.

(ii) Let $p \in (p(s, \beta \wedge \gamma), \infty)$ and $q \in (p(s, \beta \wedge \gamma), \infty]$. The *homogeneous Triebel–Lizorkin space* $\dot{F}^s_{p,q}(\mathcal{X})$ is defined by setting

$$\dot{F}^s_{p,q}(\mathcal{X}) := \{f \in (\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))' : \|f\|_{\dot{F}^s_{p,q}(\mathcal{X})} < \infty\},$$

where, for any $f \in (\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))'$,

$$\|f\|_{\dot{F}^s_{p,q}(\mathcal{X})} := \Big\|\Big[\sum_{k=-\infty}^{\infty} \delta^{-ksq} |Q_k(f)|^q\Big]^{1/q}\Big\|_{L^p(\mathcal{X})}$$

with the usual modification when $q = \infty$.

The following definition introduces the concept of Triebel–Lizorkin spaces with $p = \infty$; see [48, Definition 5.1].

DEFINITION 2.8. Let $\beta, \gamma \in (0, \eta)$, $s \in (-(\beta \wedge \gamma), \beta \wedge \gamma)$, and let $q \in (p(s, \beta \wedge \gamma), \infty]$ with $\eta$ as in Definition 2.5 and $p(s, \beta \wedge \gamma)$ as in (1.1). Let $\{Q_k\}_{k\in\mathbb{Z}}$ be an exp-ATI. For any $k \in \mathbb{Z}$ and $\alpha \in \mathcal{A}_k$, let $Q^k_\alpha$ be as in Lemma 2.4. Then the *homogeneous Triebel–Lizorkin space* $\dot{F}^s_{\infty,q}(\mathcal{X})$ is defined by setting

$$\dot{F}^s_{\infty,q}(\mathcal{X}) := \{f \in (\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))' : \|f\|_{\dot{F}^s_{\infty,q}(\mathcal{X})} < \infty\},$$

where, for any $f \in (\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))'$,

$$\|f\|_{\dot{F}^s_{\infty,q}(\mathcal{X})} := \sup_{l\in\mathbb{Z}} \sup_{\alpha\in\mathcal{A}_l} \Big[\frac{1}{\mu(Q^l_\alpha)} \int_{Q^l_\alpha} \sum_{k=l}^{\infty} \delta^{-ksq} |Q_k(f)(x)|^q \, d\mu(x)\Big]^{1/q}$$

with the usual modification when $q = \infty$.

REMARK 2.9. (i) In Definition 2.5, we need $\operatorname{diam} \mathcal{X} = \infty$ to guarantee (v). Observe that it was shown in [39, Lemma 5.1] (see also [4, Lemma 8.1]) that $\operatorname{diam} \mathcal{X} = \infty$ implies $\mu(\mathcal{X}) = \infty$. Therefore, $\operatorname{diam} \mathcal{X} = \infty$ if and only if $\mu(\mathcal{X}) = \infty$ under the assumptions of this article. Due to this, we always assume that $\mu(\mathcal{X}) = \infty$ in Sections 2 and 3.

(ii) In [48], Wang et al. proved that $\dot{B}^s_{p,q}(\mathcal{X})$ and $\dot{F}^s_{p,q}(\mathcal{X})$ in Definition 2.7 are independent of the choices of $\beta$ and $\gamma$ as in Definition 2.7, and exp-ATIs (see [48, Propositions 3.13 and 3.16] for more details). Moreover, it was also shown that $\dot{F}^s_{\infty,q}(\mathcal{X})$ in Definition 2.8 is independent of the choices of $\beta$ and $\gamma$, and exp-ATIs (see [48, Propositions 5.4 and 5.5] for more details).

Now, we introduce the concepts of $s$-gradients and $s$-Hajłasz gradients on spaces of homogeneous type (see, for instance, [37, Definition 1.1 and (2.1)]).

Definition 2.10. Let $s \in (0,\infty)$ and $u$ be a measurable function on $\mathcal{X}$.

(i) A non-negative function $g$ is called an *$s$-gradient* of $u$ if there exists a set $E \subset \mathcal{X}$ with $\mu(E) = 0$ such that, for any $x, y \in \mathcal{X} \setminus E$,

$$|u(x) - u(y)| \le [d(x,y)]^s [g(x) + g(y)].$$

Denote by $\mathcal{D}^s(u)$ the collection of all $s$-gradients of $u$.

(ii) A sequence of non-negative functions, $\{g_k\}_{k\in\mathbb{Z}}$, is called an *$s$-Hajłasz gradient* of $u$ if there exists a set $E \subset \mathcal{X}$ with $\mu(E) = 0$ such that, for any $k \in \mathbb{Z}$ and $x, y \in \mathcal{X} \setminus E$ with $\delta^{k+1} \le d(x,y) < \delta^k$,

$$|u(x) - u(y)| \le [d(x,y)]^s [g_k(x) + g_k(y)].$$

Denote by $\mathbb{D}^s(u)$ the collection of all $s$-Hajłasz gradients of $u$.

Next, we introduce the concepts of homogeneous Hajłasz–Sobolev spaces, Hajłasz–Triebel–Lizorkin spaces, and Hajłasz–Besov spaces (see, for instance, [37, Definitions 1.2 and 2.1]).

Definition 2.11. Let $s \in (0,\infty)$.

(i) Let $p \in (0,\infty)$. The *homogeneous Hajłasz–Sobolev space* $\dot{M}^{s,p}(\mathcal{X})$ is defined to be the set of all the measurable functions $u$ on $\mathcal{X}$ such that

$$\|u\|_{\dot{M}^{s,p}(\mathcal{X})} := \inf_{g\in\mathcal{D}^s(u)} \|g\|_{L^p(\mathcal{X})} < \infty.$$

(ii) Let $p \in (0,\infty)$ and $q \in (0,\infty]$. The *homogeneous Hajłasz–Triebel–Lizorkin space* $\dot{M}^s_{p,q}(\mathcal{X})$ is defined to be the set of all the measurable functions $u$ on $\mathcal{X}$ such that

$$\|u\|_{\dot{M}^s_{p,q}(\mathcal{X})} := \inf_{\{g_k\}_{k\in\mathbb{Z}}\in\mathbb{D}^s(u)} \left\| \left( \sum_{k=-\infty}^{\infty} g_k^q \right)^{1/q} \right\|_{L^p(\mathcal{X})} < \infty$$

with the usual modification when $q = \infty$.

(iii) Let $q \in (0,\infty)$. The *homogeneous Hajłasz–Triebel–Lizorkin space* $\dot{M}^s_{\infty,q}(\mathcal{X})$ is defined to be the set of all the measurable functions $u$ on $\mathcal{X}$ such that

$$\|u\|_{\dot{M}^s_{\infty,q}(\mathcal{X})} := \inf_{\{g_k\}_{k\in\mathbb{Z}}\in\mathbb{D}^s(u)} \sup_{k\in\mathbb{Z}} \sup_{x\in\mathcal{X}} \left\{ \sum_{j=k}^{\infty} \frac{1}{\mu(B(x,\delta^k))} \int_{B(x,\delta^k)} [g_j(y)]^q \, d\mu(y) \right\}^{1/q} < \infty.$$

(iv) The *homogeneous Hajłasz–Triebel–Lizorkin space* $\dot{M}^s_{\infty,\infty}(\mathcal{X})$ is defined to be the set of all the measurable functions $u$ on $\mathcal{X}$ such that

$$\|u\|_{\dot{M}^s_{\infty,\infty}(\mathcal{X})} := \inf_{\{g_k\}_{k\in\mathbb{Z}}\in\mathbb{D}^s(u)} \Big\| \sup_{k\in\mathbb{Z}} g_k \Big\|_{L^\infty(\mathcal{X})} < \infty.$$

(v) Let $p, q \in (0,\infty]$. The *homogeneous Hajłasz–Besov space* $\dot{N}^s_{p,q}(\mathcal{X})$ is defined to be the set of all the measurable functions $u$ on $\mathcal{X}$ such that

$$\|u\|_{\dot{N}^s_{p,q}(\mathcal{X})} := \inf_{\{g_k\}_{k\in\mathbb{Z}}\in\mathbb{D}^s(u)} \Big[ \sum_{k=-\infty}^{\infty} \|g_k\|^q_{L^p(\mathcal{X})} \Big]^{1/q} < \infty$$

with the usual modification when $q = \infty$.

From Definition 2.11, it is easy to get the following conclusion. We omit the details.

PROPOSITION 2.12. *Let $s \in (0,\infty)$ and $p \in (0,\infty]$. Then $\dot{M}^s_{p,\infty}(\mathcal{X}) = \dot{M}^{s,p}(\mathcal{X})$.*

Next, we recall the concept of weak lower bounds (see, for instance, [17, Definition 1.1], [49, Definition 4.4], and [1, (2) or (3)]).

DEFINITION 2.13. Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type with upper dimension $\omega$ as in (2.2). The measure $\mu$ is said to have a *weak lower bound* $Q$ with $Q \in (0, \omega]$ if there exist a positive constant $C$ and a point $x_0 \in \mathcal{X}$ such that, for any $r \in [1,\infty)$,

$$\mu(B(x_0, r)) \geq C r^Q.$$

REMARK 2.14. We point out that, in [49, Definition 4.4], $\mu$ is said to have a *lower bound* $Q$ with $Q \in (0,\omega]$ if there exists a positive constant $C$ such that, for any $x \in \mathcal{X}$ and $r \in (0,\infty)$, $\mu(B(x,r)) \geq C r^Q$. That is why we call it a weak lower bound in Definition 2.13.

As the next result illustrates, it follows from the doubling property of the measure that the weak lower bound and the lower bound conditions are equivalent when $Q = \omega$, where $\omega$ is as in (2.2).

PROPOSITION 2.15. *With $\omega$ as in (2.2), the measure $\mu$ has a weak lower bound $Q = \omega$ if and only if it has a lower bound $Q = \omega$.*

*Proof.* Clearly, the lower bound condition implies the weak lower bound condition. Now, we show the converse. To this end, suppose that the measure $\mu$ has a weak lower bound $Q = \omega$ for some fixed $x_0 \in \mathcal{X}$. Fix $x \in \mathcal{X}$ and $r \in (0,\infty)$. Next, choose an $R \in [1,\infty)$ large enough so that $R > r$ and $B(x,r) \subset B(x_0,R)$. Consider the smallest $k \in \mathbb{N}$ such that $2A_0R \leq (2A_0)^k r$, where $A_0 \in [1,\infty)$ is the constant in the quasi-triangle inequality. Note that $k \geq 1$ because $r < R$, and hence $(2A_0)^k > 1$. Also, this choice of $k$ ensures that $(2A_0)^k r \leq (2A_0)^2 R$, which further implies that $B(x_0,R) \subset$

$B(x,(2A_0)^k r)$. Using this, the weak lower bound $Q=\omega$ for the ball $B(x_0,R)$, the doubling condition in (2.1), and $(2A_0)^k r\le(2A_0)^2R$, we further conclude that

$$\begin{aligned}R^\omega &\lesssim \mu(B(x_0,R))\lesssim \mu(B(x,(2A_0)^k r))\\ &\lesssim (2A_0)^{k\omega}\mu(B(x,r))\lesssim (2A_0)^{2\omega}\left(\frac{R}{r}\right)^{\omega}\mu(B(x,r)),\end{aligned}$$

from which it follows that $\mu(B(x,r))\gtrsim r^\omega$. Thus, $\mu$ has a lower bound $Q=\omega$, as wanted. ∎

Now, we can state our main results.

THEOREM 2.16. *Let $\beta,\gamma\in(0,\eta)$ with $\eta$ as in Definition 2.5, $s\in(0,\beta\wedge\gamma)$, $p,q$ be as in Definition 2.7, and $\omega$ as in (2.2). Assume that the measure $\mu$ of $\mathcal{X}$ has a weak lower bound $Q=\omega$.*

(i) *If $p\in(\omega/(\omega+s),\infty)$ and $q\in(\omega/(\omega+s),\infty]$, then $\dot M^s_{p,q}(\mathcal{X})=\dot F^s_{p,q}(\mathcal{X})$.*
(ii) *If $p\in(\omega/(\omega+s),\infty]$ and $q\in(0,\infty]$, then $\dot N^s_{p,q}(\mathcal{X})=\dot B^s_{p,q}(\mathcal{X})$.*

**3. Relations to homogeneous grand Besov and Triebel–Lizorkin spaces.** Before we prove Theorem 2.16, we need to introduce the concepts of other important spaces, namely, the homogeneous grand Besov and Triebel–Lizorkin spaces on spaces of homogeneous type.

DEFINITION 3.1. Let $\eta$ be as in Definition 2.5, $s\in(-\eta,\eta)$, $\beta,\gamma\in(0,\eta)$, and $q\in(0,\infty]$. For any $k\in\mathbb{Z}$ and $x\in\mathcal{X}$, define

$$\mathcal{F}_k(x):=\{\phi\in\mathring{\mathcal{G}}^\eta_0(\beta,\gamma):\|\phi\|_{\mathcal{G}(x,\delta^k,\beta,\gamma)}\le 1\}.$$

(i) For any $p\in(0,\infty]$, the *homogeneous grand Besov space* $\mathcal{A}\dot B^s_{p,q}(\mathcal{X})$ is defined by setting

$$\mathcal{A}\dot B^s_{p,q}(\mathcal{X}):=\{f\in(\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))':\|f\|_{\mathcal{A}\dot B^s_{p,q}(\mathcal{X})}<\infty\},$$

where, for any $f\in(\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))'$,

$$\|f\|_{\mathcal{A}\dot B^s_{p,q}(\mathcal{X})}:=\left[\sum_{k=-\infty}^{\infty}\delta^{-ksq}\left\|\sup_{\phi\in\mathcal{F}_k(\cdot)}|\langle f,\phi\rangle|\right\|^q_{L^p(\mathcal{X})}\right]^{1/q}$$

with the usual modification when $q=\infty$.

(ii) For any $p\in(0,\infty)$, the *homogeneous grand Triebel–Lizorkin space* $\mathcal{A}\dot F^s_{p,q}(\mathcal{X})$ is defined by setting

$$\mathcal{A}\dot F^s_{p,q}(\mathcal{X}):=\{f\in(\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))':\|f\|_{\mathcal{A}\dot F^s_{p,q}(\mathcal{X})}<\infty\},$$

where, for any $f \in (\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$,

$$\|f\|_{\mathcal{A}\dot{F}^s_{p,q}(\mathcal{X})} := \left\| \left[ \sum_{k=-\infty}^{\infty} \delta^{-ksq} \sup_{\phi\in\mathcal{F}_k(\cdot)} |\langle f,\phi\rangle|^q \right]^{1/q} \right\|_{L^p(\mathcal{X})}$$

with the usual modification when $q=\infty$.

(iii) The *homogeneous grand Triebel–Lizorkin space* $\mathcal{A}\dot{F}^s_{\infty,q}(\mathcal{X})$ is defined by setting

$$\mathcal{A}\dot{F}^s_{\infty,q}(\mathcal{X}) := \{f \in (\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))' : \|f\|_{\mathcal{A}\dot{F}^s_{\infty,q}(\mathcal{X})} < \infty\},$$

where, for any $f \in (\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$,

$$\begin{aligned}\|f\|_{\mathcal{A}\dot{F}^s_{\infty,q}(\mathcal{X})} & \\ := \sup_{l\in\mathbb{Z}} \sup_{\alpha\in\mathcal{A}_l} & \left[ \frac{1}{\mu(Q_\alpha^l)} \int_{Q_\alpha^l} \sum_{k=l}^{\infty} \delta^{-ksq} \sup_{\phi\in\mathcal{F}_k(x)} |\langle f,\phi\rangle|^q \, d\mu(x) \right]^{1/q}\end{aligned}$$

with the usual modification when $q=\infty$.

Remark 3.2. Let $\{Q_k\}_{k\in\mathbb{Z}}$ be an exp-ATI. By (2.6) and (2.7), it is easy to see that, for any $k\in\mathbb{Z}$ and $x\in\mathcal{X}$, $Q_k(x,\cdot)\in\mathcal{F}_k(x)$.

We now establish the relation between homogeneous Besov and Triebel–Lizorkin spaces and homogeneous grand Besov and Triebel–Lizorkin spaces.

Theorem 3.3. *Let $\beta,\gamma\in(0,\eta)$ with $\eta$ as in Definition* 2.5, *and $s\in(-(\beta\wedge\gamma),\beta\wedge\gamma)$.*

(i) *If $p$ and $q$ are as in Definition* 2.7(ii), *then $\dot{F}^s_{p,q}(\mathcal{X}) = \mathcal{A}\dot{F}^s_{p,q}(\mathcal{X})$.*

(ii) *If $p$ and $q$ are as in Definition* 2.7(i), *then $\dot{B}^s_{p,q}(\mathcal{X}) = \mathcal{A}\dot{B}^s_{p,q}(\mathcal{X})$.*

To prove Theorem 3.3, we need several lemmas. Let us begin by recalling the following very basic inequality.

Lemma 3.4. *For any $\theta\in(0,1]$ and $\{a_j\}_{j\in\mathbb{N}}\subset\mathbb{C}$,*

$$\left(\sum_{j=1}^{\infty} |a_j|\right)^\theta \le \sum_{j=1}^{\infty} |a_j|^\theta. \tag{3.1}$$

The next lemma contains several basic and very useful estimates related to $d$ and $\mu$ on $\mathcal{X}$. One can find the details in [20, Lemma 2.1] or [24, Lemma 2.4].

Lemma 3.5. *Let $\beta,\gamma\in(0,\infty)$.*

(i) *For any $x,y\in\mathcal{X}$ and $r\in(0,\infty)$, $V(x,y)\sim V(y,x)$ and*

$$\begin{aligned}V_r(x)+V_r(y)+V(x,y) &\sim V_r(x)+V(x,y) \sim V_r(y)+V(x,y)\\ &\sim \mu(B(x,r+d(x,y))),\end{aligned}$$

*and moreover, if $d(x,y) \le r$, then $V_r(x) \sim V_r(y)$. Here the positive equivalence constants are independent of $x$, $y$, and $r$.*

(ii) *There exists a positive constant $C$ such that, for any $x_1 \in \mathcal{X}$ and $r \in (0,\infty)$,*

$$\int_{\mathcal{X}} D_\gamma(x_1, y; r)\, d\mu(y) \le C,$$

*where $D_\gamma(x_1,y;r)$ is as in (2.3).*

The following homogeneous discrete Calderón reproducing formula was obtained in [24, Theorem 5.11]. Let $j_0 \in \mathbb{N}$ be sufficiently large such that $\delta^{j_0} \le (2A_0)^{-5} C_0^{-1}$. Based on Lemma 2.4, for any $k \in \mathbb{Z}$ and $\alpha \in \mathcal{A}_k$, let

$$\mathfrak{N}(k,\alpha) := \{\tau \in \mathcal{A}_{k+j_0} : Q_\tau^{k+j_0} \subset Q_\alpha^k\}$$

and $N(k,\alpha) := \#\mathfrak{N}(k,\alpha)$. From Lemma 2.4, it follows that $N(k,\alpha) \lesssim \delta^{-j_0\omega}$ with the implicit positive constant independent of both $k$ and $\alpha$, and that $\bigcup_{\tau\in\mathfrak{N}(k,\alpha)} Q_\tau^{k+j_0} = Q_\alpha^k$. We rearrange the set $\{Q_\tau^{k+j_0} : \tau \in \mathfrak{N}(k,\alpha)\}$ as $\{Q_\alpha^{k,m}\}_{m=1}^{N(k,\alpha)}$. Also, denote by $y_\alpha^{k,m}$ an arbitrary point in $Q_\alpha^{k,m}$ and by $z_\alpha^{k,m}$ the "center" of $Q_\alpha^{k,m}$.

LEMMA 3.6. *Let $\{Q_k\}_{k=-\infty}^\infty$ be an* exp-ATI *and $\beta,\gamma \in (0,\eta)$ with $\eta$ as in Definition 2.5. For any $k \in \mathbb{Z}$, $\alpha \in \mathcal{A}_k$, and $m \in \{1,\dots,N(k,\alpha)\}$, suppose that $y_\alpha^{k,m}$ is an arbitrary point in $Q_\alpha^{k,m}$. Then there exists a sequence $\{\widetilde{Q}_k\}_{k=-\infty}^\infty$ of bounded linear integral operators on $L^2(\mathcal{X})$ such that, for any $f \in (\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$,*

$$f(\cdot) = \sum_{k=-\infty}^{\infty} \sum_{\alpha\in\mathcal{A}_k} \sum_{m=1}^{N(k,\alpha)} \mu(Q_\alpha^{k,m}) \widetilde{Q}_k(\cdot, y_\alpha^{k,m}) Q_k f(y_\alpha^{k,m})$$

*in $(\mathring{\mathcal{G}}_0^\eta(\beta,\gamma))'$. Moreover, there exists a positive constant $C$, independent of the choices of both $y_\alpha^{k,m}$, with $k \in \mathbb{Z}$, $\alpha \in \mathcal{A}_k$, and $m \in \{1,\dots,N(k,\alpha)\}$, and $f$, such that, for any $k \in \mathbb{Z}$, the kernel of $\widetilde{Q}_k$ satisfies*

(i) *for any $x, y \in \mathcal{X}$,*

$$|\widetilde{Q}_k(x,y)| \le C D_\gamma(x,y;\delta^k), \tag{3.2}$$

*where $D_\gamma(x,y;\delta^k)$ is as in (2.3);*

(ii) *for any $x, x', y \in \mathcal{X}$ with $d(x,x') \le (2A_0)^{-1}[\delta^k + d(x,y)]$,*

$$|\widetilde{Q}_k(x,y) - \widetilde{Q}_k(x',y)| \le C\left[\frac{d(x,x')}{\delta^k + d(x,y)}\right]^\beta D_\gamma(x,y;\delta^k); \tag{3.3}$$

(iii) *for any $x \in \mathcal{X}$,*

$$\int_{\mathcal{X}} \widetilde{Q}_k(x,y)\, d\mu(y) = 0 = \int_{\mathcal{X}} \widetilde{Q}_k(y,x)\, d\mu(y).$$

We also need the following three lemmas (see, for instance, [48, Lemmas 3.5 and 3.6]).

LEMMA 3.7. *Let $\gamma \in (0,\infty)$ and $p \in (\omega/(\omega+\gamma),1]$ with $\omega$ as in (2.2). Then there exists a constant $C \in [1,\infty)$ such that, for any $k,k' \in \mathbb{Z}$, $x \in \mathcal{X}$, and $y_\alpha^{k,m} \in Q_\alpha^{k,m}$ with $\alpha \in \mathcal{A}_k$ and $m \in \{1,\dots,N(k,\alpha)\}$,*

$$\begin{aligned} C^{-1}[V_{\delta^{k\wedge k'}}(x)]^{1-p} &\le \sum_{\alpha\in\mathcal{A}_k}\sum_{m=1}^{N(k,\alpha)} \mu(Q_\alpha^{k,m})[D_\gamma(x,y_\alpha^{k,m};\delta^{k\wedge k'})]^p \\ &\le C[V_{\delta^{k\wedge k'}}(x)]^{1-p}, \end{aligned}$$

*where $D_\gamma(x,y_\alpha^{k,m};\delta^{k\wedge k'})$ is as in (2.3).*

LEMMA 3.8. *Let $\gamma \in (0,\infty)$ and $r \in (\omega/(\omega+\gamma),1]$ with $\omega$ as in (2.2). Then there exists a positive constant $C$ such that, for any $k,k' \in \mathbb{Z}$, $x \in \mathcal{X}$, and $a_\alpha^{k,m} \in \mathbb{C}$ and $y_\alpha^{k,m} \in Q_\alpha^{k,m}$ with $\alpha \in \mathcal{A}_k$ and $m \in \{1,\dots,N(k,\alpha)\}$,*

$$\begin{aligned} &\sum_{\alpha\in\mathcal{A}_k}\sum_{m=1}^{N(k,\alpha)} \mu(Q_\alpha^{k,m}) D_\gamma(x,y_\alpha^{k,m};\delta^{k\wedge k'})|a_\alpha^{k,m}| \\ &\qquad \le C\delta^{[k-(k\wedge k')]\omega(1-1/r)}\Big[M\Big(\sum_{\alpha\in\mathcal{A}_k}\sum_{m=1}^{N(k,\alpha)}|a_\alpha^{k,m}|^r\mathbf{1}_{Q_\alpha^{k,m}}\Big)(x)\Big]^{1/r}, \end{aligned}$$

*where $D_\gamma(x,y_\alpha^{k,m};\delta^{k\wedge k'})$ is as in (2.3) and $M$ as in (1.2).*

The next lemma is the Fefferman–Stein vector-valued maximal inequality which was established in [13, Theorem 1.2].

LEMMA 3.9. *Let $p \in (1,\infty)$, $q \in (1,\infty]$, and $M$ be the Hardy–Littlewood maximal operator on $\mathcal{X}$ as in (1.2). Then there exists a positive constant $C$ such that, for any sequence $\{f_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathcal{X}$,*

$$\Big\|\Big\{\sum_{j\in\mathbb{Z}}[M(f_j)]^q\Big\}^{1/q}\Big\|_{L^p(\mathcal{X})} \le C\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p(\mathcal{X})}$$

*with the usual modification when $q=\infty$.*

*Proof of Theorem 3.3.* We first show (i). Assume that $f \in \mathcal{A}\dot{F}^s_{p,q}(\mathcal{X})$ and that $\{Q_k\}_{k\in\mathbb{Z}}$ is an exp-ATI. From Remark 3.2, we infer that $|Q_k(f)| \le \sup_{\phi\in\mathcal{F}_k(\cdot)}|\langle f,\phi\rangle|$ and hence $\|f\|_{\dot{F}^s_{p,q}(\mathcal{X})} \le \|f\|_{\mathcal{A}\dot{F}^s_{p,q}(\mathcal{X})}$.

Conversely, assume that $f \in \dot{F}^s_{p,q}(\mathcal{X})$. By Lemma 3.6, we know that, for any $l \in \mathbb{Z}$, $x \in \mathcal{X}$, and $\phi \in \mathcal{F}_l(x)$,

$$\langle f,\phi\rangle = \sum_{k=-\infty}^{\infty}\sum_{\alpha\in\mathcal{A}_k}\sum_{m=1}^{N(k,\alpha)} \mu(Q_\alpha^{k,m})Q_kf(y_\alpha^{k,m})\int_{\mathcal{X}}\widetilde{Q}_k(z,y_\alpha^{k,m})\phi(z)\,d\mu(z).$$

Notice that, by an argument similar to that in [48, proof of Lemma 3.9], we have, for any fixed $\eta' \in (0,\beta\wedge\gamma)$,

$$\Big|\int_{\mathcal{X}} \widetilde{Q}_k(z,y_\alpha^{k,m})\phi(z)\,d\mu(z)\Big| \lesssim \delta^{|k-l|\eta'} D_\gamma(x,y_\alpha^{k,m};\delta^{k\wedge l}),$$

where $D_\gamma(x,y_\alpha^{k,m};\delta^{k\wedge k'})$ is as in (2.3). Using this, Lemma 3.8, and the arbitrariness of $y_\alpha^{k,m}$, and choosing $r\in(\omega/(\omega+\gamma),\min\{p,q,1\})$, we obtain

$$\begin{aligned}
|\langle f,\phi\rangle| &\lesssim \sum_{k=-\infty}^{\infty} \delta^{|k-l|\eta'} \sum_{\alpha\in\mathcal{A}_k}\sum_{m=1}^{N(k,\alpha)} \mu(Q_\alpha^{k,m})|Q_kf(y_\alpha^{k,m})|D_\gamma(x,y_\alpha^{k,m};\delta^{k\wedge l})\\
&\lesssim \sum_{k=-\infty}^{\infty} \delta^{|k-l|\eta'}\delta^{[k-(k\wedge l)]\omega(1-1/r)}\\
&\quad\times\Big[M\Big(\sum_{\alpha\in\mathcal{A}_k}\sum_{m=1}^{N(k,\alpha)} |Q_kf(y_\alpha^{k,m})|^r\mathbf{1}_{Q_\alpha^{k,m}}\Big)(x)\Big]^{1/r}\\
&\lesssim \sum_{k=-\infty}^{\infty} \delta^{|k-l|\eta'}\delta^{[k-(k\wedge l)]\omega(1-1/r)}[M(|Q_kf|^r)(x)]^{1/r}.
\end{aligned}$$

Consequently, we find that

$$\begin{aligned}
\|f\|_{\mathcal{A}\dot{F}^s_{p,q}(\mathcal{X})} &\lesssim \Big\|\Big[\sum_{l=-\infty}^{\infty}\delta^{(l-k)sq}\Big\{\sum_{k=-\infty}^{\infty}\delta^{|k-l|\eta'}\delta^{[k-(k\wedge l)]\omega(1-1/r)}\\
&\qquad\qquad\times[M(\delta^{-ksr}|Q_kf|^r)]^{1/r}\Big\}^q\Big]^{1/q}\Big\|_{L^p(\mathcal{X})},
\end{aligned}$$

which, together with the Hölder inequality when $q\in(1,\infty]$, or (3.1) when $q\in(\omega/[\omega+(\beta\wedge\gamma)],1]$, implies that

$$\|f\|_{\mathcal{A}\dot{F}^s_{p,q}(\mathcal{X})} \lesssim \Big\|\Big\{\sum_{k=-\infty}^{\infty}[M(\delta^{-ksr}|Q_kf|^r)]^{q/r}\Big\}^{1/q}\Big\|_{L^p(\mathcal{X})}.$$

From this and Lemma 3.9, we deduce that

$$\|f\|_{\mathcal{A}\dot{F}^s_{p,q}(\mathcal{X})} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathcal{X})}.$$

This finishes the proof of (i).

The proof of (ii) is similar to that of (i) and we omit the details. This finishes the proof of Theorem 3.3. ∎

Next, we establish the equivalence between homogeneous Hajłasz–Besov and Hajłasz–Triebel–Lizorkin spaces, and homogeneous grand Besov and Triebel–Lizorkin spaces.

THEOREM 3.10. *Let $\beta,\gamma\in(0,\eta)$ with $\eta$ as in Definition 2.5, and $s\in(0,\beta\wedge\gamma)$. Assume that the measure $\mu$ of $\mathcal{X}$ has a weak lower bound $Q=\omega$.*

(i) *If $p\in(\omega/(\omega+s),\infty]$ and $q\in(\omega/(\omega+s),\infty]$, then $\mathcal{A}\dot{F}^s_{p,q}(\mathcal{X})=\dot{M}^s_{p,q}(\mathcal{X})$.*
(ii) *If $p\in(\omega/(\omega+s),\infty]$ and $q\in(0,\infty]$, then $\mathcal{A}\dot{B}^s_{p,q}(\mathcal{X})=\dot{N}^s_{p,q}(\mathcal{X})$.*

To prove Theorem 3.10, we need several lemmas. The following lemma was originally shown in [15, Theorem 8.7] when $s=1$ and $A_0=1$. When $s\in(0,1)$ and $A_0\in(1,\infty)$, we need more restrictions on $A_0$ and $\delta$. We borrow some ideas from [15, proof of Theorem 8.7]. In what follows, for any measurable set $E\subset\mathcal{X}$ with $\mu(E)>0$, let

$$\fint_E:=\frac{1}{\mu(E)}\int_E.$$

LEMMA 3.11. *Let $s\in(0,\infty), p\in(0,\omega/s)$, and $p^*:=\frac{\omega p}{\omega-sp}$ with $\omega$ as in (2.2). If $A_0\delta^{p/\omega}<1$, then there exists a positive constant $C$ such that, for any $B_0:=B(x_0,r_0)\subset\mathcal{X}$ with $x_0\in\mathcal{X}$ and $r_0\in(0,\infty)$, $u\in\dot{M}^{s,p}(B(x_0,\delta^{-1}r_0))$, and $g\in\mathcal{D}^s(u)$, one has $u\in L^{p^*}(B_0)$ and*

$$\inf_{c\in\mathbb{R}}\left[\fint_{B_0}|u(y)-c|^{p^*}\,d\mu(y)\right]^{1/p^*}\le Cr_0^s\left\{\fint_{\delta^{-1}B_0}[g(y)]^p\,d\mu(y)\right\}^{1/p}. \tag{3.4}$$

*Proof.* If $\int_{\delta^{-1}B_0}[g(y)]^p\,d\mu(y)=\infty$, then (3.4) holds true. If

$$\int_{\delta^{-1}B_0}[g(y)]^p\,d\mu(y)=0,$$

then we know that $g(x)=0$ for almost every $x\in\delta^{-1}B_0$ and hence there exists a $c\in\mathbb{R}$ such that $u(x)=c$ for almost every $x\in\delta^{-1}B_0$. Thus, in this case, (3.4) holds true.

In what follows, we assume that

$$0<\int_{\delta^{-1}B_0}[g(y)]^p\,d\mu(y)<\infty.$$

Note that this implies $g>0$ almost everywhere on $\mathcal{X}$. Moreover, we may also assume that, for every $x\in\delta^{-1}B_0$,

$$g(x)\ge\delta^{1+1/p}\left\{\fint_{\delta^{-1}B_0}[g(y)]^p\,d\mu(y)\right\}^{1/p}, \tag{3.5}$$

as otherwise we may replace $g$ by $\widetilde{g}(x):=g(x)+\{\fint_{\delta^{-1}B_0}[g(y)]^p\,d\mu(y)\}^{1/p}$ for any $x\in\delta^{-1}B_0$, because

$$\left\{\fint_{\delta^{-1}B_0}[\widetilde{g}(y)]^p\,d\mu(y)\right\}^{1/p}\lesssim\left\{\fint_{\delta^{-1}B_0}[g(y)]^p\,d\mu(y)\right\}^{1/p}.$$

For any $k \in \mathbb{Z}$, define

$$E_k := \{x \in \delta^{-1}B_0 : g(x) \le \delta^{-k}\}.$$

It is easy to see that, for any $k \in \mathbb{Z}$, we have $E_{k-1} \subset E_k$ and

$$\lim_{k\to\infty} \mu(E_k) = \mu(\delta^{-1}B_0). \tag{3.6}$$

Since $g > 0$ almost everywhere on $\mathcal{X}$, we also have

$$\mu\Big(\delta^{-1}B_0 \setminus \bigcup_{k\in\mathbb{Z}} [E_k \setminus E_{k-1}]\Big) = 0,$$

which allows us to write

$$\int_{\delta^{-1}B_0} [g(y)]^p \, d\mu(y) \sim \sum_{k=-\infty}^{\infty} \delta^{-kp} \mu(E_k \setminus E_{k-1}). \tag{3.7}$$

For any $c \in \mathbb{R}$, if we let $a_k := \sup_{y\in B_0\cap E_k} |u(y) - c|$, then $a_k$ is non-decreasing and

$$\int_{B_0} |u(y) - c|^{p^*} \, d\mu(y) \le \sum_{k=-\infty}^{\infty} a_k^{p^*} \mu(B_0 \cap [E_k \setminus E_{k-1}]). \tag{3.8}$$

Note that, if $\mu(\delta^{-1}B_0 \setminus E_{k-1}) = 0$, then $\mu(E_k \setminus E_{k-1}) = 0$. Thus, to estimate both (3.7) and (3.8), we only need to consider $k \in \mathbb{Z}$ such that $\mu(\delta^{-1}B_0 \setminus E_{k-1}) > 0$, which is always assumed in what follows. Let

$$b := (4A_0)^{-\omega} \delta^{\omega} r_0^{-\omega} \mu(\delta^{-1}B_0) \tag{3.9}$$

and

$$r_k := 2b^{-1/\omega} [\mu(\delta^{-1}B_0 \setminus E_{k-1})]^{1/\omega}.$$

Then we know that $r_k \in (0, \infty)$. Moreover, from the Chebyshev inequality, we infer that

$$\begin{aligned} \mu(\delta^{-1}B_0 \setminus E_k) &= \mu(\{x \in \delta^{-1}B_0 : g(x) > \delta^{-k}\}) \\ &\le \delta^{kp} \int_{\delta^{-1}B_0} [g(y)]^p \, d\mu(y), \end{aligned} \tag{3.10}$$

which implies that $\lim_{k\to\infty} r_k = 0$. Thus, there exists a $k_0 \in \mathbb{Z}$, which will be determined later, such that, for any $k > k_0$, we can find an $x_k \in B_0$ satisfying $B(x_k, r_k) \subset \delta^{-1}B_0$, where $r_k \le \delta^{-1}r_0$. Observe that, by the doubling condition of $\mathcal{X}$, we can conclude that, for any $k > k_0$, $\mu(B(x_k, r_k)) \ge br_k^{\omega}$. Combining this and the definition of $r_k$, we find that

$$\mu(B(x_k, r_k)) \ge br_k^{\omega} > \mu(\delta^{-1}B_0 \setminus E_{k-1}) = \mu(\delta^{-1}B_0) - \mu(E_{k-1}).$$

From this, we deduce that $B(x_k, r_k) \cap E_{k-1} \ne \emptyset$, that is, there exists an $x_{k-1} \in B(x_k, r_k) \cap E_{k-1}$. Now, if $B(x_{k-1}, r_{k-1}) \subset \delta^{-1}B_0$, then we can repeat the above procedure to find an $x_{k-2}$ such that $x_{k-2} \in B(x_{k-1}, r_{k-1}) \cap E_{k-2}$. In summary, for any $i \in \{1, \dots, k - k_0 + 1\}$, if $B(x_{k-i}, r_{k-i}) \subset \delta^{-1}B_0$, then we can find an $x_{k-i-1}$ such that $x_{k-i-1} \in B(x_{k-i}, r_{k-i}) \cap E_{k-i-1}$. We now want to determine $k_0$. Note that, by (3.10), $x_k \in B_0$, and the assumption that $A_0\delta^{p/\omega} < 1$, we find that, for any $y \in B(x_{k_0}, r_{k_0})$,

$$\begin{aligned} d(y,x_0) &\le A_0[d(y,x_k)+d(x_k,x_0)] \\ &< A_0 d(y,x_k) + A_0 r_0 \\ &\le A_0^2[d(y,x_{k_0})+d(x_{k_0},x_k)] + A_0 r_0 \\ &\le A_0^2 r_{k_0} + A_0^3[d(x_{k_0},x_{k_0+1})+d(x_{k_0+1},x_k)] + A_0 r_0 \\ &\le A_0^2 r_{k_0} + A_0^3 r_{k_0+1} + \cdots + A_0^{k-k_0+1} r_{k-1} + A_0^{k-k_0+1} r_k + A_0 r_0 \\ &\le 2A_0^{-k_0+3} b^{-1/\omega}\Big\{\int_{\delta^{-1}B_0}[g(z)]^p\,d\mu(z)\Big\}^{1/\omega}\sum_{i=k_0-1}^{k-1}(A_0\delta^{p/\omega})^i + A_0 r_0 \\ &\le A_0^2\delta^{(k_0-1)p/\omega}\frac{2b^{-1/\omega}}{1-A_0\delta^{p/\omega}}\Big\{\int_{\delta^{-1}B_0}[g(z)]^p\,d\mu(z)\Big\}^{1/\omega} + A_0 r_0. \end{aligned}$$

If

$$A_0^2\delta^{(k_0-1)p/\omega}\frac{2b^{-1/\omega}}{1-A_0\delta^{p/\omega}}\Big\{\int_{\delta^{-1}B_0}[g(z)]^p\,d\mu(z)\Big\}^{1/\omega} + A_0 r_0 \le \delta^{-1} r_0, \tag{3.11}$$

then we conclude that, for any $i \in \{1,\dots,k-k_0\}$, $B(x_{k-i}, r_{k-i}) \subset \delta^{-1}B_0$. Observe that (3.11) is equivalent to

$$\begin{aligned} \delta^{1-k_0} \ge &\left[\frac{2A_0^2}{(1-A_0\delta^{p/\omega})(\delta^{-1}-A_0)}\right]^{\omega/p} \\ &\times (br_0^\omega)^{-1/p}\Big\{\int_{\delta^{-1}B_0}[g(z)]^p\,d\mu(z)\Big\}^{1/p}. \end{aligned} \tag{3.12}$$

We claim that, if (3.12) holds true, then $r_k \le \delta^{-1}r_0$ for any $k \ge k_0$. Indeed, from the definition of $r_k$, (3.10), (3.12), and the fact that $\delta$ is very small, we deduce that

$$\begin{aligned} r_k &\le 2b^{-1/\omega}\Big\{\delta^{(k-1)p}\int_{\delta^{-1}B_0}[g(y)]^p\,d\mu(y)\Big\}^{1/\omega} \\ &\le 2b^{-1/\omega}\delta^{(k_0-1)p/\omega}\Big\{\int_{\delta^{-1}B_0}[g(y)]^p\,d\mu(y)\Big\}^{1/\omega} \\ &\le 2b^{-1/\omega}\left[\frac{2A_0^2}{(1-A_0\delta^{p/\omega})(\delta^{-1}-A_0)}\right]^{-1} b^{1/\omega} r_0\Big\{\int_{\delta^{-1}B_0}[g(y)]^p\,d\mu(y)\Big\}^{-1/\omega} \\ &\le \frac{(1-A_0\delta^{p/\omega})(\delta^{-1}-A_0)}{A_0^2} r_0 \le \delta^{-1} r_0. \end{aligned}$$

Thus, the above claim holds true.

Observe that (3.5) implies $E_k = \emptyset$ for $k \in \mathbb{Z}$ small enough. Using this and (3.6), we conclude that there exists a $\widetilde{k}_0 \in \mathbb{Z}$ such that

$$\mu(E_{\widetilde{k}_0-1}) < \delta\mu(\delta^{-1}B_0) \le \mu(E_{\widetilde{k}_0}). \tag{3.13}$$

From this, we deduce that $E_{\widetilde{k}_0} \neq \emptyset$ and, by (3.5), for any $x \in E_{\widetilde{k}_0}$ we have

$$\delta^{1+1/p}\left\{ \fint_{\delta^{-1}B_0} [g(y)]^p \, d\mu(y) \right\}^{1/p} \le g(x) \le \delta^{-\widetilde{k}_0}.$$

On the other hand, since $\delta$ is very small, we may assume that $\delta < 1/2$. Then, by both (3.13) and (3.10), we know that

$$\begin{aligned}\delta\mu(\delta^{-1}B_0) &< (1-\delta)\mu(\delta^{-1}B_0) < \mu(\delta^{-1}B_0 \setminus E_{\widetilde{k}_0-1}) \\ &\le \delta^{(\widetilde{k}_0-1)p} \int_{\delta^{-1}B_0} [g(y)]^p \, d\mu(y).\end{aligned}$$

Combining the above two estimates, we find that

$$\delta^{1+1/p}\left\{ \fint_{\delta^{-1}B_0} [g(y)]^p \, d\mu(y) \right\}^{1/p} \le \delta^{-\widetilde{k}_0} \le \delta^{-1-1/p}\left\{ \fint_{\delta^{-1}B_0} [g(y)]^p \, d\mu(y) \right\}^{1/p}.$$

Let $l_0$ be the smallest integer such that

$$\delta^{-l_0} > \max\left\{ \delta^{-2-1/p}\left[ \frac{2A_0^2}{(1-A_0\delta^{p/\omega})(\delta^{-1}-A_0)} \right]^{\omega/p}, 1 \right\} \left[ \frac{\mu(\delta^{-1}B_0)}{br_0^\omega} \right]^{1/p}$$

and let $k_0 := \widetilde{k}_0 + l_0$. Then we conclude that (3.12) holds true and

$$\begin{aligned}\delta^{-k_0} &= \delta^{-1}\delta^{-\widetilde{k}_0}\delta^{-(l_0-1)} \\ &\le \delta^{-1}\delta^{-1-1/p}\left\{ \fint_{\delta^{-1}B_0} [g(y)]^p \, d\mu(y) \right\}^{1/p} \\ &\quad \times \max\left\{ \delta^{-2-1/p}\left[ \frac{2A_0^2}{(1-A_0\delta^{p/\omega})(\delta^{-1}-A_0)} \right]^{\omega/p}, 1 \right\} \left[ \frac{\mu(\delta^{-1}B_0)}{br_0^\omega} \right]^{1/p} \\ &\lesssim (br_0^\omega)^{-1/p}\left\{ \int_{\delta^{-1}B_0} [g(z)]^p \, d\mu(z) \right\}^{1/p},\end{aligned}$$

which, together with (3.12), implies that

$$\delta^{-k_0} \sim (br_0^\omega)^{-1/p}\left\{ \int_{\delta^{-1}B_0} [g(z)]^p \, d\mu(z) \right\}^{1/p}. \tag{3.14}$$

Now, we estimate $a_k$ by considering two cases of $k$.

Case 1: $k > k_0$. In this case, it suffices to consider $k > k_0$ such that $E_k \cap B_0 \neq \emptyset$. For any $x_k \in E_k \cap B_0$, choose $\{x_{k-1}, \dots, x_{k_0}\}$ as above. Then, from $g \in \mathcal{D}^s(u)$, the definition of $r_k$, (3.10), and $p \in (0, \omega/s)$, we find that, for any $c \in \mathbb{R}$,

$$
\begin{aligned}
&|u(x_k) - c| \\
&\quad \le \sum_{i=0}^{k-k_0-1} |u(x_{k-i}) - u(x_{k-i-1})| + |u(x_{k_0}) - c| \\
&\quad \le \sum_{i=0}^{k-k_0-1} [d(x_{k-i}, x_{k-i-1})]^s [g(x_{k-i}) + g(x_{k-i-1})] + |u(x_{k_0}) - c| \\
&\quad \lesssim \sum_{i=0}^{k-k_0-1} \delta^{-k+i} r_{k-i}^s + |u(x_{k_0}) - c| \\
&\quad \lesssim b^{-s/\omega} \Big\{ \int_{\delta^{-1}B_0} [g(z)]^p \, d\mu(z) \Big\}^{s/\omega} \sum_{i=0}^{k-k_0-1} \delta^{-k+i} \delta^{(k-i-1)ps/\omega} + |u(x_{k_0}) - c| \\
&\quad \lesssim b^{-s/\omega} \Big\{ \int_{\delta^{-1}B_0} [g(z)]^p \, d\mu(z) \Big\}^{s/\omega} \delta^{-(k-1)(1-ps/\omega)} + |u(x_{k_0}) - c|,
\end{aligned}
$$

which implies that

$$
a_k \lesssim b^{-s/\omega} \Big\{ \int_{\delta^{-1}B_0} [g(z)]^p \, d\mu(z) \Big\}^{s/\omega} \delta^{-k(1-ps/\omega)} + \sup_{x \in E_{k_0}} |u(x) - c|.
$$

Choose a $\widetilde{c} \in \mathbb{R}$ such that $\operatorname{ess\,inf}_{x \in E_{k_0}} |u(x) - \widetilde{c}| = 0$. Then we can find $\{y_j\}_{j \in \mathbb{N}} \subset E_{k_0}$ such that $\lim_{j \to \infty} |u(y_j) - \widetilde{c}| = 0$. As $g \in \mathcal{D}^s(u)$, using the definition of $E_{k_0}$, we have, for any $x \in E_{k_0}$,

$$
\begin{aligned}
(3.15) \qquad |u(x) - \widetilde{c}\,| &= \lim_{j \to \infty} |[u(x) - \widetilde{c}] - [u(y_j) - \widetilde{c}]| = \lim_{j \to \infty} |u(x) - u(y_j)| \\
&\le \varlimsup_{j \to \infty} [d(x, y_j)]^s [g(x) + g(y_j)] \le 2^{s+1} A_0^s r_0^s \delta^{-k_0 - s},
\end{aligned}
$$

which further implies that, for any $k > k_0$,

$$
(3.16) \qquad a_k \lesssim b^{-s/\omega} \Big\{ \int_{\delta^{-1}B_0} [g(z)]^p \, d\mu(z) \Big\}^{s/\omega} \delta^{-k(1-ps/\omega)} + r_0^s \delta^{-k_0}.
$$

Case 2: $k \le k_0$. In this case, by both (3.15) and the fact that $E_k$ is increasing, we find that

$$
\begin{aligned}
(3.17) \qquad a_k &= \sup_{y \in B_0 \cap E_k} |u(y) - \widetilde{c}| \\
&\le \sup_{y \in B_0 \cap E_{k_0}} |u(y) - \widetilde{c}| \le \sup_{y \in E_{k_0}} |u(y) - \widetilde{c}| \lesssim r_0^s \delta^{-k_0},
\end{aligned}
$$

where we let $a_k := 0$ if $B_0 \cap E_k = \emptyset$.

From (3.16), (3.17), (3.8), (3.7), (3.14), and (3.9), we deduce that

$$\begin{aligned}
&\int_{B_0} |u(y)-\widetilde{c}|^{p^*}\,d\mu(y)\\
&\qquad\le \sum_{k=-\infty}^{\infty} a_k^{p^*}\mu(B_0\cap[E_k\setminus E_{k-1}])\\
&\qquad\lesssim b^{-sp^*/\omega}\Big\{\int_{\delta^{-1}B_0}[g(z)]^p\,d\mu(z)\Big\}^{sp^*/\omega}\sum_{k=-\infty}^{\infty}\delta^{-k(1-ps/\omega)p^*}\mu(E_k\setminus E_{k-1})\\
&\qquad\quad + r_0^{sp^*}\delta^{-k_0p^*}\mu(B_0)\\
&\qquad\lesssim b^{-sp^*/\omega}\Big\{\int_{\delta^{-1}B_0}[g(z)]^p\,d\mu(z)\Big\}^{sp^*/\omega}\Big\{\int_{\delta^{-1}B_0}[g(z)]^p\,d\mu(z)\Big\}\\
&\qquad\quad + \mu(B_0)r_0^{sp^*}(br_0^{\omega})^{-p^*/p}\Big\{\int_{\delta^{-1}B_0}[g(z)]^p\,d\mu(z)\Big\}^{p^*/p}\\
&\qquad\lesssim \Big[1+\frac{\mu(B_0)}{br_0^{\omega}}\Big]b^{-sp^*/\omega}\Big\{\int_{\delta^{-1}B_0}[g(z)]^p\,d\mu(z)\Big\}^{p^*/p}\\
&\qquad\lesssim \frac{\mu(B_0)}{br_0^{\omega}}b^{-sp^*/\omega}\Big\{\int_{\delta^{-1}B_0}[g(z)]^p\,d\mu(z)\Big\}^{p^*/p},
\end{aligned}$$

which implies that

$$\Big[\fint_{B_0}|u(y)-\widetilde{c}|^{p^*}\,d\mu(y)\Big]^{1/p^*}\lesssim (br_0^{\omega})^{-1/p^*}b^{-s/\omega}\Big\{\int_{\delta^{-1}B_0}[g(y)]^p\,d\mu(y)\Big\}^{1/p}.$$

Recalling that $b=(4A_0)^{-\omega}\delta^{\omega}r_0^{-\omega}\mu(\delta^{-1}B_0)$, we then conclude that

$$(br_0^{\omega})^{-1/p^*}b^{-s/\omega}\lesssim r_0^s[\mu(\delta^{-1}B_0)]^{-1/p}.$$

This finishes the proof of Lemma 3.11. ∎

REMARK 3.12. Let $p^*, u$, and $B_0$ be as in Lemma 3.11. If $p^*\in[1,\infty)$, then $u\in L^1(B_0)$, and moreover the left hand side of (3.4) can be replaced by

$$\Big[\fint_{B_0}|u(y)-u_{B_0}|^{p^*}\,d\mu(y)\Big]^{1/p^*},$$

where

$$u_{B_0}:=\fint_{B_0}u(y)\,d\mu(y).$$

Indeed, we can find a $c_0 \in \mathbb{R}$ such that

$$\Big[\fint_{B_0} |u(y) - c_0|^{p^*}\, d\mu(y)\Big]^{1/p^*} \le 2 \inf_{c\in\mathbb{R}} \Big[\fint_{B_0} |u(y) - c|^{p^*}\, d\mu(y)\Big]^{1/p^*}.$$

By the Hölder inequality, we have

$$\begin{aligned}\Big[\fint_{B_0} |u(y) - u_{B_0}|^{p^*}\, d\mu(y)\Big]^{1/p^*} &= \Big[\fint_{B_0} \Big|u(y) - \fint_{B_0} u(z)\, d\mu(z)\Big|^{p^*} d\mu(y)\Big]^{1/p^*} \\ &\le \Big[\fint_{B_0}\fint_{B_0} |u(y) - u(z)|^{p^*}\, d\mu(z)\, d\mu(y)\Big]^{1/p^*} \\ &\lesssim \Big[\fint_{B_0} |u(y) - c_0|^{p^*}\, d\mu(y)\Big]^{1/p^*}.\end{aligned}$$

This finishes the proof of the above claim.

The next result is a consequence of Lemma 3.11 and highlights the fact that functions in $\dot{M}^s_{p,q}(\mathcal{X})$ are actually locally integrable whenever $p \in (\omega/(\omega+s),\infty)$.

Corollary 3.13. *Let $s \in (0,\infty)$, $q \in (0,\infty]$, and $p \in (\omega/(\omega+s),\infty)$ with $\omega$ as in* (2.2). *Then every function in $\dot{M}^s_{p,q}(\mathcal{X})$ is locally integrable on $\mathcal{X}$.*

*Proof.* Fix a $u \in \dot{M}^s_{p,q}(\mathcal{X})$ and observe that, if $\{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^s(u)$, then $g \in \mathcal{D}^s(u)$, where

$$g := \Big(\sum_{k=-\infty}^{\infty} g_k^q\Big)^{1/q}$$

with the usual modification when $q = \infty$. Thus, $u \in \dot{M}^{s,p}(\mathcal{X})$. Consider any ball $B_0 \subset \mathcal{X}$ and suppose $A_0\delta^{p/\omega} < 1$. If we choose a $t \in (\omega/(\omega+s), p \wedge (\omega/s))$, then $u \in \dot{M}^{s,t}(\delta^{-1}B_0)$ and Lemma 3.11 implies $u \in L^{t^*}(B_0)$, where $t^* = \frac{\omega t}{\omega - st} > 1$. Thus, $u \in L^1(B_0)$, which completes the proof. ∎

The following result also follows from Lemma 3.11.

Corollary 3.14. *Let $s \in (0,\infty)$, $p \in (\omega/(\omega+s), \omega/s)$, and $p^* := \frac{\omega p}{\omega - sp}$ with $\omega$ as in* (2.2). *Assume that $\mathcal{X}$ has a weak lower bound $Q = \omega$. Then, for any $u \in \dot{M}^{s,p}(\mathcal{X})$, there exists a constant $C \in \mathbb{R}$ such that $u - C \in L^{p^*}(\mathcal{X})$ and*

$$\|u - C\|_{L^{p^*}(\mathcal{X})} \le \widetilde{C}\|u\|_{\dot{M}^{s,p}(\mathcal{X})}, \tag{3.18}$$

*where $\widetilde{C}$ is a positive constant independent of $u$.*

*Proof.* Let $u \in \dot{M}^{s,p}(\mathcal{X})$ and fix a point $x_0 \in \mathcal{X}$. For any $k \in \mathbb{N}$, let $B_k := B(x_0, k)$. Choose a $g \in \mathcal{D}^s(u)$ such that $\|g\|_{L^p(\mathcal{X})} \le 2\|u\|_{\dot{M}^{s,p}(\mathcal{X})}$. By

Lemma 3.11, we find that, for any $k \in \mathbb{N}$,

$$\Big[\fint_{B_k} |u(y) - u_{B_k}|^{p^*}\, d\mu(y)\Big]^{1/p^*} \lesssim k^s \Big\{\fint_{\delta^{-1}B_k} [g(y)]^p\, d\mu(y)\Big\}^{1/p} \lesssim k^s [\mu(B_k)]^{-1/p} \|u\|_{\dot{M}^{s,p}(\mathcal{X})}.$$

This, together with the assumption that $\mathcal{X}$ has a weak lower bound $Q = \omega$, and Proposition 2.15, implies that

$$\Big[\int_{B_k} |u(y) - u_{B_k}|^{p^*}\, d\mu(y)\Big]^{1/p^*} \lesssim k^s [\mu(B_k)]^{1/p^* - 1/p} \|u\|_{\dot{M}^{s,p}(\mathcal{X})} \lesssim k^{s(1-Q/\omega)} \|u\|_{\dot{M}^{s,p}(\mathcal{X})} \lesssim \|u\|_{\dot{M}^{s,p}(\mathcal{X})}. \tag{3.19}$$

From this, we find that, for any $k \in \mathbb{N}$,

$$\begin{aligned} |u_{B_k} - u_{B_1}| &\le \frac{1}{\mu(B_1)} \int_{B_1} |u(y) - u_{B_k}|\, d\mu(y) \\ &\le \frac{1}{[\mu(B_1)]^{1/p^*}} \Big[\int_{B_1} |u(y) - u_{B_k}|^{p^*}\, d\mu(y)\Big]^{1/p^*} \\ &\le \frac{1}{[\mu(B_1)]^{1/p^*}} \|u\|_{\dot{M}^{s,p}(\mathcal{X})}, \end{aligned}$$

which implies that $\{u_{B_k}\}_{k\in\mathbb{N}} \subset \mathbb{R}$ is a bounded sequence. Therefore, there exist a subsequence $\{u_{B_{k_j}}\}_{j\in\mathbb{N}}$ and a constant $C \in \mathbb{R}$ such that $C = \lim_{j\to\infty} u_{B_{k_j}}$. Moreover, by (3.19) and the Fatou lemma, we further conclude that

$$\begin{aligned} &\Big[\int_{\mathcal{X}} |u(x) - C|^{p^*}\, d\mu(x)\Big]^{1/p^*} \\ &\quad = \Big[\int_{\mathcal{X}} \lim_{j\to\infty} |[u(x) - u_{B_{k_j}}] \mathbf{1}_{B_{k_j}}(x)|^{p^*}\, d\mu(x)\Big]^{1/p^*} \\ &\quad \le \varliminf_{j\to\infty} \Big[\int_{B_{k_j}} |u(x) - u_{B_{k_j}}|^{p^*}\, d\mu(x)\Big]^{1/p^*} \lesssim \|u\|_{\dot{M}^{s,p}(\mathcal{X})}. \end{aligned}$$

This finishes the proof of Corollary 3.14. ∎

Remark 3.15. Let $\omega$ be as in (2.2) and $p \in (0, \omega)$. In [1, Theorem 22], Alvarado et al. proved that, if $\mathcal{X}$ is uniformly perfect (see [1, (39)]), then (3.18) with $s = 1$ is equivalent to $\mathcal{X}$ having a lower bound.

The following lemma is a Poincaré type inequality for $\mathbb{D}^s(u)$ (see also [37, Lemma 2.1]).

Lemma 3.16. *Let $s \in (0, \infty)$. Then there exists a positive constant $C$ such that, for any $k \in \mathbb{Z}$, any measurable function $u$ on $\mathcal{X}$, any $x \in \mathcal{X}$, and* *any* $\{g_j\}_{j\in\mathbb{Z}} \in \mathbb{D}^s(u)$,

$$\inf_{c\in\mathbb{R}} \fint_{B(x,\delta^k)} |u(y)-c|\,d\mu(y) \le C\delta^{ks}\sum_{j=k-3}^{k-1}\fint_{B(x,\delta^{k-2})} g_j(y)\,d\mu(y). \tag{3.20}$$

*Proof.* Observe that, for any $k\in\mathbb{Z}$ and $x\in\mathcal{X}$,

$$\begin{aligned}\inf_{c\in\mathbb{R}} &\fint_{B(x,\delta^k)} |u(y)-c|\,d\mu(y) \\ &\le \fint_{B(x,\delta^k)} |u(y)-u_{B(x,\delta^{k-2})\setminus B(x,A_0\delta^{k-1})}|\,d\mu(y) \\ &\le \fint_{B(x,\delta^k)}\fint_{B(x,\delta^{k-2})\setminus B(x,A_0\delta^{k-1})} |u(y)-u(z)|\,d\mu(y)\,d\mu(z).\end{aligned} \tag{3.21}$$

As $\delta$ is very small, for any $y\in B(x,\delta^k)$ and $z\in B(x,\delta^{k-2})\setminus B(x,A_0\delta^{k-1})$ we have

$$\begin{aligned} d(y,z) &\le A_0[d(y,x)+d(x,z)] < 2A_0\delta^{k-2} \le \delta^{k-3},\\ d(x,z) &\le A_0[d(x,y)+d(y,z)] < A_0\delta^k + A_0 d(y,z),\end{aligned}$$

which implies that

$$\delta^k \le d(y,z) < \delta^{k-3}.$$

From this, we deduce that there exists a unique $j_0\in\{k-1,k-2,k-3\}$ such that

$$\delta^{j_0+1}\le d(y,z)<\delta^{j_0}$$

and hence

$$|u(y)-u(z)| \le [d(y,z)]^s[g_{j_0}(y)+g_{j_0}(z)] \le \delta^{(k-3)s}\sum_{j=k-3}^{k-1}[g_j(y)+g_j(z)].$$

Therefore, by (3.21), we conclude that

$$\begin{aligned}\inf_{c\in\mathbb{R}} &\fint_{B(x,\delta^k)} |u(y)-c|\,d\mu(y)\\ &\lesssim \delta^{ks}\sum_{j=k-3}^{k-1}\fint_{B(x,\delta^k)}\fint_{B(x,\delta^{k-2})\setminus B(x,A_0\delta^{k-1})} [g_j(y)+g_j(z)]\,d\mu(y)\,\mu(z)\\ &\lesssim \delta^{ks}\sum_{j=k-3}^{k-1}\fint_{B(x,\delta^{k-2})} g_j(y)\,d\mu(y),\end{aligned}$$

which completes the proof of Lemma 3.16. ∎

REMARK 3.17. Similarly to Remark 3.12, under the same assumptions as in Lemma 3.16, the left hand side of (3.20) can be replaced by

$$\fint_{B(x,\delta^k)} |u(y) - u_{B(x,\delta^k)}| \, d\mu(y).$$

Using Lemma 3.11, we can show the following Poincaré type inequality which is very useful in the case when $p \in (0,1]$.

LEMMA 3.18. *Let $s \in (0,\infty)$, $p \in (0,1]$, and $\varepsilon, \varepsilon' \in (0,s)$ with $\varepsilon < \varepsilon'$. If $A_0\delta^{p/\omega} < 1$, then there exists a positive constant $C$ such that, for any $k \in \mathbb{Z}$, any $x \in \mathcal{X}$, any measurable function $u$, and any $\{g_j\}_{j\in\mathbb{Z}} \in \mathbb{D}^s(u)$,*

$$\inf_{c\in\mathbb{R}} \Big[ \fint_{B(x,\delta^k)} |u(y) - c|^{\frac{\omega p}{\omega - \varepsilon p}} \, d\mu(y) \Big]^{\frac{\omega - \varepsilon p}{\omega p}} \leq C\delta^{k\varepsilon'} \sum_{j=k-2}^{\infty} \delta^{j(s-\varepsilon')} \Big\{ \fint_{B(x,\delta^{k-1})} [g_j(y)]^p \, d\mu(y) \Big\}^{1/p}. \tag{3.22}$$

*Proof.* Without loss of generality, we may assume that the right hand side of (3.22) is finite. For any $k \in \mathbb{Z}$ and $x \in \mathcal{X}$, let

$$g(x) := \Big\{ \sum_{j=k-2}^{\infty} \delta^{j(s-\varepsilon)p} [g_j(x)]^p \Big\}^{1/p}.$$

We claim that $g \in \mathcal{D}^s(u)$ and $u \in \dot{M}^{\varepsilon,p}(B(x,\delta^{k-1}))$. Indeed, for any $y, z \in B(x,\delta^{k-1})$, we have

$$d(y,z) \leq A_0[d(y,x) + d(x,z)] < 2A_0\delta^{k-1} < \delta^{k-2}.$$

Therefore, there exists a unique integer $j_0 \geq k-2$ such that

$$\delta^{j_0+1} \leq d(y,z) < \delta^{j_0}.$$

Then, since $\{g_j\}_{j\in\mathbb{Z}} \in \mathbb{D}^s(u)$, it follows that there exists an $E \subset \mathcal{X}$ with $\mu(E) = 0$ such that, for any $y, z \in B(x,\delta^{k-1}) \setminus E$,

$$\begin{aligned} |u(y) - u(z)| &\leq [d(y,z)]^s [g_{j_0}(y) + g_{j_0}(z)] \leq [d(y,z)]^{\varepsilon} \delta^{j_0(s-\varepsilon)} [g_{j_0}(y) + g_{j_0}(z)] \\ &\leq [d(y,z)]^{\varepsilon} [g(y) + g(z)], \end{aligned}$$

which implies $g \in \mathcal{D}^{\varepsilon}(u)$. On the other hand, since $p \in (0,1]$ and $0 < \varepsilon < \varepsilon' < s$, by the Hölder inequality with exponent $1/p \geq 1$ we conclude that

$$
\begin{aligned}
&\|g\|_{L^p(B(x,\delta^{k-1}))}\\
&\quad= \Big\{ \int_{B(x,\delta^{k-1})} \sum_{j=k-2}^{\infty} \delta^{j(s-\varepsilon)p}[g_j(y)]^p \, d\mu(y) \Big\}^{1/p}\\
&\quad= \Big\{ \sum_{j=k-2}^{\infty} \delta^{j(\varepsilon'-\varepsilon)p}\delta^{j(s-\varepsilon')p} \int_{B(x,\delta^{k-1})} [g_j(y)]^p \, d\mu(y) \Big\}^{1/p}\\
&\quad\leq \Big[ \sum_{j=k-2}^{\infty} \delta^{j(\varepsilon'-\varepsilon)p(1/p)'} \Big]^{\frac{1}{(1/p)'p}} \sum_{j=k-2}^{\infty} \delta^{j(s-\varepsilon')} \Big\{ \int_{B(x,\delta^{k-1})} [g_j(y)]^p \, d\mu(y) \Big\}^{1/p}\\
&\quad\lesssim \delta^{k(\varepsilon'-\varepsilon)}[V_{\delta^{k-1}}(x)]^{1/p} \sum_{j=k-2}^{\infty} \delta^{j(s-\varepsilon')} \Big\{ \fint_{B(x,\delta^{k-1})} [g_j(y)]^p \, d\mu(y) \Big\}^{1/p} < \infty.
\end{aligned}
$$

From this, we deduce that $u \in \dot{M}^{\varepsilon,p}(B(x,\delta^{k-1}))$. Combining the above claim and Lemma 3.11, and using the Hölder inequality with exponent $1/p \geq 1$, we find that

$$
\begin{aligned}
&\inf_{c\in\mathbb{R}} \Big[ \fint_{B(x,\delta^k)} |u(y)-c|^{\frac{\omega p}{\omega-\varepsilon p}} \, d\mu(y) \Big]^{\frac{\omega-\varepsilon p}{\omega p}}\\
&\quad\lesssim \delta^{k\varepsilon} \Big\{ \fint_{B(x,\delta^{k-1})} [g(y)]^p \, d\mu(y) \Big\}^{1/p}\\
&\quad\sim \delta^{k\varepsilon} \Big\{ \sum_{j=k-2}^{\infty} \delta^{j(s-\varepsilon)p} \fint_{B(x,\delta^{k-1})} [g_j(y)]^p \, d\mu(y) \Big\}^{1/p}\\
&\quad\lesssim \delta^{k\varepsilon} \Big\{ \sum_{j=k-2}^{\infty} \delta^{j(\varepsilon'-\varepsilon)p(1/p)'} \Big\}^{\frac{1}{(1/p)'p}} \sum_{j=k-2}^{\infty} \delta^{j(s-\varepsilon')} \Big\{ \fint_{B(x,\delta^{k-1})} [g_j(y)]^p \, d\mu(y) \Big\}^{1/p}\\
&\quad\lesssim \delta^{k\varepsilon'} \sum_{j=k-2}^{\infty} \delta^{j(s-\varepsilon')} \Big\{ \fint_{B(x,\delta^{k-1})} [g_j(y)]^p \, d\mu(y) \Big\}^{1/p}.
\end{aligned}
$$

This finishes the proof of Lemma 3.18. ∎

The following lemma illustrates that any element of $\mathcal{A}\dot{F}^s_{p,\infty}(\mathcal{X})$ is a locally integrable function.

LEMMA 3.19. *Let $\beta, \gamma \in (0, \eta)$ with $\eta$ as in Definition 2.5, $s \in (0, \beta \wedge \gamma)$, and $p \in (\omega/(\omega+s), \infty)$ with $\omega$ as in (2.2). Assume that the measure $\mu$ of $\mathcal{X}$ has a weak lower bound $Q = \omega$. Then, for any $f \in \mathcal{A}\dot{F}^s_{p,\infty}(\mathcal{X})$, there exists an $\widetilde{f} \in L^1_{\mathrm{loc}}(\mathcal{X})$ such that $f = \widetilde{f}$ in $(\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))'$.*

To prove Lemma 3.19, we need the concept of approximations of the identity with exponential decay and integration 1; see [23, Definition 2.8] for more details.

Definition 3.20. Let $\eta \in (0,1)$ be as in Definition 2.5. A sequence $\{Q_k\}_{k\in\mathbb{Z}}$ of bounded linear integral operators on $L^2(\mathcal{X})$ is called an *approximation of the identity with exponential decay and integration* 1 (for short, 1-exp-ATI) if $\{Q_k\}_{k\in\mathbb{Z}}$ has the following properties:

(i) for any $k \in \mathbb{Z}$, $Q_k$ satisfies (ii), (iii), and (iv) of Definition 2.5 but without the decay factor
$$\exp\left\{-\nu\left[\frac{\max\{d(x,\mathcal{Y}^k), d(y,\mathcal{Y}^k)\}}{\delta^k}\right]^a\right\};$$

(ii) for any $k \in \mathbb{Z}$ and $x \in \mathcal{X}$,
$$\int_{\mathcal{X}} Q_k(x,y)\,d\mu(y) = 1 = \int_{\mathcal{X}} Q_k(y,x)\,d\mu(y);$$

(iii) if $P_k := Q_k - Q_{k-1}$ for any $k \in \mathbb{Z}$, then $\{P_k\}_{k\in\mathbb{Z}}$ is an exp-ATI.

Remark 3.21. As was pointed out in [23, Remark 2.9], the existence of a 1-exp-ATI is guaranteed by [4, Lemma 10.1]. Moreover, by the proofs of both [24, Proposition 2.9] and [20, Proposition 2.7(iv)], we know that, if $\{Q_k\}_{k\in\mathbb{Z}}$ is a 1-exp-ATI, then, for any $f \in L^2(\mathcal{X})$, $\lim_{k\to\infty} Q_k f = f$ in $L^2(\mathcal{X})$.

By an argument similar to that in [24, proof of Proposition 2.10], we have the following conclusion; we omit the details.

Lemma 3.22. *Let $\beta,\gamma \in (0,\eta)$ with $\eta$ as in Definition* 2.5, *$s \in (0,\beta\wedge\gamma)$, $k \in \mathbb{Z}$, $x,y \in \mathcal{X}$, and $\{Q_k\}_{k\in\mathbb{Z}}$ be a* 1-exp-ATI. *For any $z \in \mathcal{X}$, let*
$$\phi(z) := \delta^{ks}[d(x,y)]^{-s}[Q_k(x,z) - Q_k(y,z)].$$
*If $d(x,y) \in (0,\delta^k]$, then $\phi \in \mathcal{F}_k(x)$, where $\mathcal{F}_k(x)$ is as in Definition* 3.1.

*Proof of Lemma 3.19.* Assume that $f \in \mathcal{A}\dot{F}^s_{p,\infty}(\mathcal{X})$ and that $\{Q_k\}_{k\in\mathbb{Z}}$ is a 1-exp-ATI. For any $x \in \mathcal{X}$, let
$$g(x) := \sup_{k\in\mathbb{Z}} \delta^{-ks} \sup_{\phi\in\mathcal{F}_k(x)} |\langle f,\phi\rangle|.$$
First, for any $k \in \mathbb{Z}$, $i \in \mathbb{N}$, and $x \in \mathcal{X}$, we have
$$\begin{aligned}|Q_k f(x) - Q_{k+i} f(x)| &\le \sum_{j=0}^{i-1} |Q_{k+j} f(x) - Q_{k+j+1} f(x)|\\ &= \sum_{j=0}^{i-1} |\langle f, Q_{k+j}(x,\cdot) - Q_{k+j+1}(x,\cdot)\rangle|.\end{aligned}$$

Note that, by (2.6) and (2.7), we obtain, for any $k \in \mathbb{Z}$ and $x \in \mathcal{X}$,

$$Q_{k+j}(x,\cdot) - Q_{k+j+1}(x,\cdot) \in \mathcal{F}_{k+j+1}(x).$$

From this, we deduce that

$$|Q_k f(x) - Q_{k+i} f(x)| \le \sum_{j=0}^{i-1} \delta^{(k+j+1)s} g(x) \lesssim \delta^{ks} g(x). \tag{3.23}$$

We begin with the case $p \in (1,\infty)$. To this end, note that (3.23) implies that $\{Q_k f - Q_{k+i} f\}_{i\in\mathbb{N}}$ is a Cauchy sequence of $L^p(\mathcal{X})$. By the completeness of $L^p(\mathcal{X})$, there exists an $f_k \in L^p(\mathcal{X})$ such that

$$\lim_{i\to\infty} (Q_k f - Q_{k+i} f) = f_k$$

both in $L^p(\mathcal{X})$ and pointwise. Observe that, for any $k, k' \in \mathbb{Z}$, we have

$$\begin{aligned} f_k &= \lim_{i\to\infty} (Q_k f - Q_{k+i} f) = Q_k f - Q_{k'} f + \lim_{i\to\infty} (Q_{k'} f - Q_{k+i} f) \\ &= Q_k f - Q_{k'} f + f_{k'} \end{aligned}$$

both in $L^p(\mathcal{X})$ and pointwise. Let $\widetilde{f} := Q_0 f - f_0$. Then $\widetilde{f} \in L^1_{\mathrm{loc}}(\mathcal{X})$ and, for any $k \in \mathbb{Z}$, $\widetilde{f} = Q_k f - f_k$. On the other hand, as $Q_k f \to f$ in $(\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))'$ as $k \to \infty$, it follows that, for any $\psi \in \mathring{\mathcal{G}}^\eta_0(\beta,\gamma)$,

$$\begin{aligned} \langle \widetilde{f}, \psi \rangle &= \int_{\mathcal{X}} \widetilde{f}(x)\psi(x)\,d\mu(x) \\ &= \int_{\mathcal{X}} \Big\{ Q_0 f(x) - \lim_{i\to\infty} [Q_0 f(x) - Q_i f(x)] \Big\} \psi(x)\,d\mu(x) \\ &= \int_{\mathcal{X}} Q_0 f(x)\psi(x)\,d\mu(x) - \lim_{i\to\infty} \int_{\mathcal{X}} [Q_0 f(x) - Q_i f(x)]\psi(x)\,d\mu(x) \\ &= \lim_{i\to\infty} \int_{\mathcal{X}} Q_i f(x)\psi(x)\,d\mu(x) = \lim_{i\to\infty} \langle Q_i f, \psi \rangle = \langle f, \psi \rangle, \end{aligned}$$

which completes the proof of the lemma in the case when $p \in (1,\infty)$.

Next, we consider the case $p \in (\omega/(\omega+s), 1]$. For any $x, y \in \mathcal{X}$, let $k_0 \in \mathbb{Z}$ be such that $\delta^{k_0+1} < d(x,y) \le \delta^{k_0}$. Then, by Lemma 3.22 and (3.23), we conclude that, for any $k \in \mathbb{Z}$ with $k > k_0$,

$$\begin{aligned} |Q_k f(x) - Q_k f(y)| &\le |Q_k f(x) - Q_{k_0} f(x)| + |Q_{k_0} f(x) - Q_{k_0} f(y)| \\ &\quad + |Q_{k_0} f(y) - Q_k f(y)| \\ &\lesssim \delta^{k_0 s}[g(x) + g(y)] \lesssim [d(x,y)]^s [g(x) + g(y)]. \end{aligned} \tag{3.24}$$

On the other hand, for any $k \in \mathbb{Z}$ with $k \le k_0$,

$$|Q_k f(x) - Q_k f(y)| = [d(x,y)]^s \delta^{-ks} \langle f, \delta^{ks}[d(x,y)]^{-s}[Q_k(x,\cdot) - Q_k(y,\cdot)] \rangle.$$

Note that, due to $k \le k_0$, we have $d(x,y) \le \delta^k$. From this and Lemma 3.22, we deduce that

$$\delta^{ks}[d(x,y)]^{-s}[Q_k(x,\cdot) - Q_k(y,\cdot)] \in \mathcal{F}_k(x),$$

which implies that

$$|Q_k f(x) - Q_k f(y)| \lesssim [d(x,y)]^s [g(x) + g(y)].$$

Combining this and (3.24), we conclude that, for any $k \in \mathbb{Z}$,

$$|Q_k f(x) - Q_k f(y)| \lesssim [d(x,y)]^s [g(x) + g(y)]. \tag{3.25}$$

Moreover, as $f \in \mathcal{A}\dot{F}^s_{p,\infty}(\mathcal{X})$, we know that $g \in L^p(\mathcal{X})$. From this and (3.25), we further infer that, for any $k \in \mathbb{Z}$,

$$Q_k f \in \dot{M}^{s,p}(\mathcal{X}) \quad \text{and} \quad \|Q_k f\|_{\dot{M}^{s,p}(\mathcal{X})} \lesssim \|g\|_{L^p(\mathcal{X})} \sim \|f\|_{\mathcal{A}\dot{F}^s_{p,\infty}(\mathcal{X})}.$$

Using this and Corollary 3.14, we find that, for any $k \in \mathbb{Z}$, there exists a constant $C_k$ such that

$$Q_k f - C_k \in L^{p^*}(\mathcal{X}) \quad \text{and} \quad \|Q_k f - C_k\|_{L^{p^*}(\mathcal{X})} \lesssim \|Q_k f\|_{\dot{M}^{s,p}(\mathcal{X})} \lesssim \|f\|_{\mathcal{A}\dot{F}^s_{p,\infty}(\mathcal{X})}.$$

From this and the weak compactness property of $L^{p^*}(\mathcal{X})$ (recall that $p^* > 1$ in this case), we deduce that there exist a subsequence $\{Q_{k_j} f - C_{k_j}\}_{j\in\mathbb{N}}$ and a function $\widetilde{f} \in L^{p^*}(\mathcal{X})$ such that

$$\widetilde{f} = \lim_{j\to\infty} [Q_{k_j} f - C_{k_j}]$$

both weakly in $L^{p^*}(\mathcal{X})$ and also in $(\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))'$. Moreover, by (3.23), we find that $Q_{k_j} f - Q_{k_{j'}} f \in L^p(\mathcal{X})$ and $[Q_{k_j} f - C_{k_j}] - [Q_{k_{j'}} f - C_{k_{j'}}] \in L^{p^*}(\mathcal{X})$ for any $j, j' \in \mathbb{N}$, which further implies that $C_{k_j} = C_{k_{j'}}$. Since $Q_{k_j} f \to f$ in $(\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))'$ as $j \to \infty$, we conclude that

$$f = f - C_{k_0} = \lim_{j\to\infty} [Q_{k_j} f - C_{k_j}] = \widetilde{f}$$

in $(\mathring{\mathcal{G}}^\eta_0(\beta,\gamma))'$. This finishes the proof of Lemma 3.19. ■

*Proof of Theorem 3.10.* We only prove (i); the proof of (ii) is similar. We first show

$$\dot{M}^s_{p,q}(\mathcal{X}) \subset \mathcal{A}\dot{F}^s_{p,q}(\mathcal{X}).$$

To this end, fix a $u \in \dot{M}^s_{p,q}(\mathcal{X})$ and recall that $u$ is locally integrable on $\mathcal{X}$ by Corollary 3.13. With this in mind, we consider five cases of $p$ and $q$.

CASE 1: $p \in (1,\infty)$ and $q \in (1,\infty]$. In this case, we only consider $q \in (1,\infty)$, because the proof for $q = \infty$ is similar. Choose a $\{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^s(u)$ such that

$$\left\| \left( \sum_{k=-\infty}^{\infty} g_k^q \right)^{1/q} \right\|_{L^p(\mathcal{X})} \lesssim \|u\|_{\dot{M}^s_{p,q}(\mathcal{X})}.$$

Then, for any $k \in \mathbb{Z}$, $x \in \mathcal{X}$, and $\phi \in \mathcal{F}_k(x)$,

$$
\begin{aligned}
|\langle u, \phi\rangle| &= \left|\int_{\mathcal{X}} \phi(y)u(y)\,d\mu(y)\right| = \left|\int_{\mathcal{X}} \phi(y)[u(y) - u_{B(x,\delta^k)}]\,d\mu(y)\right| \\
&\le \int_{\mathcal{X}} D_\gamma(x,y;\delta^k)|u(y) - u_{B(x,\delta^k)}|\,d\mu(y) \\
&\lesssim \sum_{j=0}^{\infty} \delta^{j\gamma} \fint_{B(x,\delta^{k-j})} |u(y) - u_{B(x,\delta^k)}|\,d\mu(y),
\end{aligned}
$$

where $D_\gamma(x,y;\delta^k)$ is as in (2.3). Notice that, for any $k \in \mathbb{Z}$, $j \in \mathbb{Z}_+$, and $x \in \mathcal{X}$,

$$
\begin{aligned}
&\fint_{B(x,\delta^{k-j})} |u(y) - u_{B(x,\delta^k)}|\,d\mu(y) \\
&\quad = \fint_{B(x,\delta^{k-j})} |u(y) - u_{B(x,\delta^{k-1})} + u_{B(x,\delta^{k-1})} - u_{B(x,\delta^k)}|\,d\mu(y) \\
&\quad \le \fint_{B(x,\delta^{k-j})} |u(y) - u_{B(x,\delta^{k-1})}|\,d\mu(y) \\
&\qquad + \frac{1}{V_{\delta^k}(x)} \int_{B(x,\delta^k)} |u(y) - u_{B(x,\delta^{k-1})}|\,d\mu(y) \\
&\quad \lesssim \fint_{B(x,\delta^{k-j})} |u(y) - u_{B(x,\delta^{k-1})}|\,d\mu(y) \\
&\qquad + \fint_{B(x,\delta^{k-1})} |u(y) - u_{B(x,\delta^{k-1})}|\,d\mu(y) \\
&\quad \lesssim \sum_{i=0}^{j} \fint_{B(x,\delta^{k-i})} |u(y) - u_{B(x,\delta^{k-i})}|\,d\mu(y).
\end{aligned}
$$

Combining the above two inequalities, we obtain, for any $k \in \mathbb{Z}$,

$$
\begin{aligned}
|\langle u, \phi\rangle| &\lesssim \sum_{j=0}^{\infty} \delta^{j\gamma} \sum_{i=0}^{j} \fint_{B(x,\delta^{k-i})} |u(y) - u_{B(x,\delta^{k-i})}|\,d\mu(y) \\
&\sim \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} \delta^{j\gamma} \fint_{B(x,\delta^{k-i})} |u(y) - u_{B(x,\delta^{k-i})}|\,d\mu(y) \\
&\lesssim \sum_{i=0}^{\infty} \delta^{i\gamma} \fint_{B(x,\delta^{k-i})} |u(y) - u_{B(x,\delta^{k-i})}|\,d\mu(y).
\end{aligned} \tag{3.26}
$$

From this and Lemma 3.16 (see also Remark 3.17), we get, for any $k \in \mathbb{Z}$,

$$|\langle u, \phi\rangle| \lesssim \sum_{i=0}^{\infty} \delta^{i\gamma}\delta^{(k-i)s} \sum_{j=k-i-3}^{k-i-1} \fint_{B(x,\delta^{k-i-2})} g_j(y)\,d\mu(y) \tag{3.27}$$
$$\lesssim \delta^{ks} \sum_{j=-\infty}^{k} \sum_{i=k-j-3}^{k-j-1} \delta^{i(\gamma-s)} \fint_{B(x,\delta^{k-i-2})} g_j(y)\,d\mu(y)$$
$$\lesssim \delta^{k\gamma} \sum_{j=-\infty}^{k} \delta^{-j(\gamma-s)} \fint_{B(x,\delta^{j-2})} g_j(y)\,d\mu(y)$$
$$\lesssim \delta^{k\gamma} \sum_{j=-\infty}^{k} \delta^{-j(\gamma-s)} M(g_j)(x).$$

By the Hölder inequality and Lemma 3.9, we conclude that

$$\|u\|_{\mathcal{A}\dot{F}^s_{p,q}(\mathcal{X})} \lesssim \left\|\left\{\sum_{k=-\infty}^{\infty} \delta^{-ksq}\left[\delta^{k\gamma} \sum_{j=-\infty}^{k} \delta^{-j(\gamma-s)} M(g_j)\right]^q\right\}^{1/q}\right\|_{L^p(\mathcal{X})}$$
$$\lesssim \left\|\left\{\sum_{k=-\infty}^{\infty} \delta^{k(\gamma-s)} \sum_{j=-\infty}^{k} \delta^{-j(\gamma-s)} [M(g_j)]^q\right\}^{1/q}\right\|_{L^p(\mathcal{X})}$$
$$\lesssim \left\|\left\{\sum_{j=-\infty}^{\infty} [M(g_j)]^q\right\}^{1/q}\right\|_{L^p(\mathcal{X})} \lesssim \left\|\left\{\sum_{j=-\infty}^{\infty} g_j^q\right\}^{1/q}\right\|_{L^p(\mathcal{X})}$$
$$\lesssim \|u\|_{\dot{M}^s_{p,q}(\mathcal{X})},$$

which is the desired estimate in this case.

CASE 2: $p \in (1,\infty)$ and $q \in (\omega/(\omega+s), 1]$. In this case, choose a $\{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^s(u)$ as in Case 1. Recall that we may assume $A_0\delta^{p/\omega} < 1$. Hence, combining (3.26) and Lemma 3.18 [with $p = \omega/(\omega+\varepsilon) < 1$], we find that, for any fixed $\varepsilon, \varepsilon' \in (0, s)$ with $\varepsilon < \varepsilon'$,

$$|\langle u, \phi\rangle| \lesssim \sum_{i=0}^{\infty} \delta^{i\gamma}\delta^{(k-i)\varepsilon'} \sum_{j=k-i-2}^{\infty} \delta^{j(s-\varepsilon')} \tag{3.28}$$
$$\times \left\{\fint_{B(x,\delta^{k-i-1})} [g_j(y)]^{\frac{\omega}{\omega+\varepsilon}}\,d\mu(y)\right\}^{\frac{\omega+\varepsilon}{\omega}}$$
$$\lesssim \delta^{k\varepsilon'} \sum_{j=-\infty}^{k-2} \delta^{j(s-\varepsilon')} \left[M\left([g_j(x)]^{\frac{\omega}{\omega+\varepsilon}}\right)\right]^{\frac{\omega+\varepsilon}{\omega}} \sum_{i=k-j-2}^{\infty} \delta^{i(\gamma-\varepsilon')}$$
$$+ \delta^{k\varepsilon'} \sum_{j=k-1}^{\infty} \delta^{j(s-\varepsilon')} \left[M\left([g_j(x)]^{\frac{\omega}{\omega+\varepsilon}}\right)\right]^{\frac{\omega+\varepsilon}{\omega}} \sum_{i=0}^{\infty} \delta^{i(\gamma-\varepsilon')}$$

$$\lesssim \delta^{k\gamma} \sum_{j=-\infty}^{k-2} \delta^{j(s-\gamma)} \big[M\big([g_j(x)]^{\frac{\omega}{\omega+\varepsilon}}\big)\big]^{\frac{\omega+\varepsilon}{\omega}} + \delta^{k\varepsilon'} \sum_{j=k-1}^{\infty} \delta^{j(s-\varepsilon')} \big[M\big([g_j(x)]^{\frac{\omega}{\omega+\varepsilon}}\big)\big]^{\frac{\omega+\varepsilon}{\omega}}.$$

Choosing an $\varepsilon \in (0, s)$ such that $\frac{\omega}{\omega+s} < \frac{\omega}{\omega+\varepsilon} < q$, and using (3.28), (3.1) (with $\theta = q$), and Lemma 3.9, we conclude that

$$\begin{aligned}
\|u\|_{\mathcal{A}\dot{F}^s_{p,q}(\mathcal{X})} &\lesssim \bigg\|\bigg( \sum_{k=-\infty}^{\infty} \delta^{-ksq} \bigg\{\delta^{k\gamma} \sum_{j=-\infty}^{k-2} \delta^{j(s-\gamma)} [M([g_j]^{\frac{\omega}{\omega+\varepsilon}})]^{\frac{\omega+\varepsilon}{\omega}} \\
&\qquad + \delta^{k\varepsilon'} \sum_{j=k-1}^{\infty} \delta^{j(s-\varepsilon')} \big[M\big([g_j]^{\frac{\omega}{\omega+\varepsilon}}\big)\big]^{\frac{\omega+\varepsilon}{\omega}} \bigg\}^q\bigg)^{1/q}\bigg\|_{L^p(\mathcal{X})} \\
&\lesssim \bigg\|\bigg\{ \sum_{k=-\infty}^{\infty} \delta^{-ksq}\delta^{k\gamma q} \sum_{j=-\infty}^{k-2} \delta^{j(s-\gamma)q} \big[M\big([g_j]^{\frac{\omega}{\omega+\varepsilon}}\big)\big]^{\frac{(\omega+\varepsilon)q}{\omega}} \bigg\}^{1/q}\bigg\|_{L^p(\mathcal{X})} \\
&\qquad + \bigg\|\bigg\{ \sum_{k=-\infty}^{\infty} \delta^{-ksq}\delta^{k\varepsilon' q} \sum_{j=k-1}^{\infty} \delta^{j(s-\varepsilon')q} \big[M\big([g_j]^{\frac{\omega}{\omega+\varepsilon}}\big)\big]^{\frac{(\omega+\varepsilon)q}{\omega}} \bigg\}^{1/q}\bigg\|_{L^p(\mathcal{X})} \\
&\lesssim \bigg\|\bigg\{ \sum_{j=-\infty}^{\infty} \big[M\big([g_j]^{\frac{\omega}{\omega+\varepsilon}}\big)\big]^{\frac{(\omega+\varepsilon)q}{\omega}} \bigg\}^{1/q}\bigg\|_{L^p(\mathcal{X})} \lesssim \bigg\|\bigg\{ \sum_{j=-\infty}^{\infty} g_j^q \bigg\}^{1/q}\bigg\|_{L^p(\mathcal{X})} \\
&\lesssim \|u\|_{\dot{M}^s_{p,q}(\mathcal{X})}.
\end{aligned}$$

This is the desired estimate in this case.

CASE 3: $p \in (\omega/(\omega+s), 1]$ and $q \in (\omega/(\omega+s), \infty]$. In this case, the proof is similar to that in Case 2; the details are omitted.

CASE 4: $p = \infty$ and $q \in (1, \infty]$. In this case, we only consider $q \in (1, \infty)$ because the proof for $q = \infty$ is similar. Choose a $\{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^s(u)$ such that

$$\sup_{k\in\mathbb{Z}} \sup_{x\in\mathcal{X}} \bigg\{ \sum_{j=k}^{\infty} \fint_{B(x,\delta^k)} [g_j(y)]^q \, d\mu(y) \bigg\}^{1/q} \lesssim \|u\|_{\dot{M}^s_{\infty,q}(\mathcal{X})}. \tag{3.29}$$

From (3.27) and the Hölder inequality, we infer that, for any $l \in \mathbb{Z}$ and $x \in \mathcal{X}$,

$$\begin{aligned}
&\fint_{B(x,2A_0C_0\delta^l)} \sum_{k=l}^{\infty} \delta^{-ksq} \sup_{\phi\in\mathcal{F}_k(z)} |\langle u, \phi\rangle|^q \, d\mu(z) \\
&\quad \lesssim \fint_{B(x,2A_0C_0\delta^l)} \sum_{k=l}^{\infty} \delta^{-ksq} \bigg[\delta^{k\gamma} \sum_{j=-\infty}^{k} \delta^{-j(\gamma-s)} \fint_{B(z,\delta^{j-2})} g_j(y) \, d\mu(y)\bigg]^q \, d\mu(z)
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \fint_{B(x,2A_0C_0\delta^l)} \sum_{k=l}^{\infty} \delta^{k(\gamma-s)} \sum_{j=-\infty}^{k} \delta^{-j(\gamma-s)} \Big[\fint_{B(z,\delta^{j-2})} g_j(y)\,d\mu(y)\Big]^q \,d\mu(z)\\
&\lesssim \fint_{B(x,2A_0C_0\delta^l)} \sum_{k=l}^{\infty} \delta^{k(\gamma-s)} \sum_{j=-\infty}^{l-1} \delta^{-j(\gamma-s)} \Big[\fint_{B(z,\delta^{j-2})} g_j(y)\,d\mu(y)\Big]^q \,d\mu(z)\\
&\quad + \fint_{B(x,2A_0C_0\delta^l)} \sum_{k=l}^{\infty} \delta^{k(\gamma-s)} \sum_{j=l}^{k} \delta^{-j(\gamma-s)} \Big[\fint_{B(z,\delta^{j-2})} g_j(y)\,d\mu(y)\Big]^q \,d\mu(z)\\
&=: \mathrm{Y}_1 + \mathrm{Y}_2.
\end{aligned}
$$

To estimate $\mathrm{Y}_1$, by the Hölder inequality with exponent $q \in (1,\infty)$, and by (3.29), we find that, for any $j \in \mathbb{Z}$ and $z \in \mathcal{X}$,

$$
\Big[\fint_{B(z,\delta^{j-2})} g_j(y)\,d\mu(y)\Big]^q \le \fint_{B(z,\delta^{j-2})} [g_j(y)]^q\,d\mu(y) \lesssim \|u\|^q_{\dot{M}^s_{\infty,q}(\mathcal{X})}.
$$

From this, we further deduce that

$$
\mathrm{Y}_1 \lesssim \|u\|^q_{\dot{M}^s_{\infty,q}(\mathcal{X})} \fint_{B(x,2A_0C_0\delta^l)} \sum_{k=l}^{\infty} \delta^{k(\gamma-s)} \sum_{j=-\infty}^{l-1} \delta^{-j(\gamma-s)}\,d\mu(z) \lesssim \|u\|^q_{\dot{M}^s_{\infty,q}(\mathcal{X})}.
$$

To estimate $\mathrm{Y}_2$, note that, for any $j \in \mathbb{Z}$ with $j \ge l$, $z \in B(x, 2A_0C_0\delta^l)$, and $y \in B(z,\delta^{j-2})$, we have

$$
\begin{aligned}
d(y,x) &\le A_0[d(y,z)+d(z,x)] < A_0\delta^{j-2} + 2A_0^2C_0\delta^l\\
&\le (A_0\delta + 2A_0^2C_0\delta^3)\delta^{l-3} \le \delta^{l-3},
\end{aligned}
$$

which further implies that

$$
\begin{aligned}
\mathrm{Y}_2 &\lesssim \fint_{B(x,2A_0C_0\delta^l)} \sum_{k=l}^{\infty} \delta^{k(\gamma-s)} \sum_{j=l}^{k} \delta^{-j(\gamma-s)}\\
&\qquad \times \Big[\fint_{B(z,\delta^{j-2})} g_j(y)\mathbf{1}_{B(x,\delta^{l-3})}(y)\,d\mu(y)\Big]^q \,d\mu(z)\\
&\lesssim \fint_{B(x,2A_0C_0\delta^l)} \sum_{k=l}^{\infty} \delta^{k(\gamma-s)} \sum_{j=l}^{k} \delta^{-j(\gamma-s)} [M(g_j\mathbf{1}_{B(x,\delta^{l-3})})(z)]^q\,d\mu(z)\\
&\lesssim \sum_{j=l}^{\infty} \fint_{B(x,2A_0C_0\delta^l)} [M(g_j\mathbf{1}_{B(x,\delta^{l-3})})(z)]^q\,d\mu(z)\\
&\lesssim \sum_{j=l}^{\infty} \fint_{B(x,2A_0C_0\delta^l)} [g_j(z)\mathbf{1}_{B(x,\delta^{l-3})}(z)]^q\,d\mu(z)
\end{aligned}
$$

$$\lesssim \sum_{j=l-3}^{\infty} \fint_{B(x,\delta^{l-3})} [g_j(z)]^q \, d\mu(z) \lesssim \|u\|^q_{\dot{M}^s_{\infty,q}(\mathcal{X})}.$$

Combining the estimates of both $\mathrm{Y}_1$ and $\mathrm{Y}_2$, we conclude that, for any $l \in \mathbb{Z}$ and $x \in \mathcal{X}$,

$$\fint_{B(x,2A_0C_0\delta^l)} \sum_{k=l}^{\infty} \delta^{-ksq} \sup_{\phi\in\mathcal{F}_k(z)} |\langle u, \phi\rangle|^q \, d\mu(z) \lesssim \|u\|^q_{\dot{M}^s_{\infty,q}(\mathcal{X})},$$

which, together with Lemma 2.4(iii), implies the desired estimate.

Case 5: $p=\infty$ and $q\in(\omega/(\omega+s),1]$. In this case, choose a $\{g_k\}_{k\in\mathbb{Z}}\in\mathbb{D}^s(u)$ as in Case 4. Arguing as in (3.28), we have, for any fixed $\varepsilon,\varepsilon'\in(0,s)$ with $\varepsilon<\varepsilon'$,

$$\begin{aligned}
&\fint_{B(x,2A_0C_0\delta^l)} \sum_{k=l}^{\infty} \delta^{-ksq} \sup_{\phi\in\mathcal{F}_k(z)} |\langle u, \phi\rangle|^q \, d\mu(z)\\
&\quad\lesssim \fint_{B(x,2A_0C_0\delta^l)} \sum_{k=l}^{\infty} \delta^{-ksq} \sum_{i=0}^{\infty} \delta^{i\gamma q}\delta^{(k-i)\varepsilon' q}\\
&\qquad\times \sum_{j=k-i-2}^{\infty} \delta^{j(s-\varepsilon')q}\left\{\fint_{B(z,\delta^{k-i-1})} [g_j(y)]^{\frac{\omega}{\omega+\varepsilon}} \, d\mu(y)\right\}^{\frac{(\omega+\varepsilon)q}{\omega}} d\mu(z)\\
&\quad\sim \fint_{B(x,2A_0C_0\delta^l)} \sum_{k=l}^{\infty} \delta^{-ksq} \sum_{m=-\infty}^{k} \delta^{(k-m)\gamma q}\delta^{m\varepsilon' q}\\
&\qquad\times \sum_{j=m-2}^{\infty} \delta^{j(s-\varepsilon')q}\left\{\fint_{B(z,\delta^{m-1})} [g_j(y)]^{\frac{\omega}{\omega+\varepsilon}} \, d\mu(y)\right\}^{\frac{(\omega+\varepsilon)q}{\omega}} d\mu(z)\\
&\quad\lesssim \fint_{B(x,2A_0C_0\delta^l)} \sum_{m=-\infty}^{\infty} \delta^{(l\vee m)(\gamma-s)q}\delta^{m(\varepsilon'-\gamma)q}\\
&\qquad\times \sum_{j=m-2}^{\infty} \delta^{j(s-\varepsilon')q}\left\{\fint_{B(z,\delta^{m-1})} [g_j(y)]^{\frac{\omega}{\omega+\varepsilon}} \, d\mu(y)\right\}^{\frac{(\omega+\varepsilon)q}{\omega}} d\mu(z)\\
&\quad\lesssim \fint_{B(x,2A_0C_0\delta^l)} \sum_{m=-\infty}^{l} \delta^{l(\gamma-s)q}\delta^{m(\varepsilon'-\gamma)q}\\
&\qquad\times \sum_{j=m-2}^{\infty} \delta^{j(s-\varepsilon')q}\left\{\fint_{B(z,\delta^{m-1})} [g_j(y)]^{\frac{\omega}{\omega+\varepsilon}} \, d\mu(y)\right\}^{\frac{(\omega+\varepsilon)q}{\omega}} d\mu(z)\\
&\qquad+ \fint_{B(x,2A_0C_0\delta^l)} \sum_{m=l+1}^{\infty} \delta^{m(\varepsilon'-s)q}\dots\\
&\quad=: \mathrm{Y}_3+\mathrm{Y}_4.
\end{aligned}$$

To estimate $\mathrm{Y}_3$, choosing an $\varepsilon \in (0,s)$ such that $\omega/(\omega+s) < \omega/(\omega+\varepsilon) < q$, and using the Hölder inequality and (3.29), we find that, for any $m \in \mathbb{Z}$ and $z \in \mathcal{X}$,

$$\left\{ \fint_{B(z,\delta^{m-1})} [g_j(y)]^{\frac{\omega}{\omega+\varepsilon}} \, d\mu(y) \right\}^{\frac{(\omega+\varepsilon)q}{\omega}} \le \fint_{B(z,\delta^{m-1})} [g_j(y)]^q \, d\mu(y) \lesssim \|u\|^q_{\dot{M}^s_{\infty,q}(\mathcal{X})},$$

which implies that

$$\mathrm{Y}_3 \lesssim \|u\|^q_{\dot{M}^s_{\infty,q}(\mathcal{X})} \sum_{m=-\infty}^{l} \delta^{l(\gamma-s)q} \delta^{m(\varepsilon'-\gamma)q} \sum_{j=m-2}^{\infty} \delta^{j(s-\varepsilon')q} \lesssim \|u\|^q_{\dot{M}^s_{\infty,q}(\mathcal{X})}.$$

To estimate $\mathrm{Y}_4$, we first observe that, for any $m \in \mathbb{Z}$ with $m \ge l+1$, $z \in B(x, 2A_0C_0\delta^l)$, and $y \in B(z, \delta^{m-1})$,

$$\begin{aligned} d(y,x) &\le A_0[d(y,z) + d(z,x)] < A_0\delta^{m-2} + 2A_0^2C_0\delta^l \\ &\le (A_0\delta + 2A_0^2C_0\delta^2)\delta^{l-2} \le \delta^{l-2}, \end{aligned}$$

which further implies that

$$\begin{aligned} \mathrm{Y}_4 &\lesssim \fint_{B(x,2A_0C_0\delta^l)} \sum_{m=l+1}^{\infty} \delta^{m(\varepsilon'-s)q} \sum_{j=m-2}^{\infty} \delta^{j(s-\varepsilon')q} \\ &\quad \times \left\{ \fint_{B(z,\delta^{m-1})} [g_j(y)\mathbf{1}_{B(x,\delta^{l-2})}(y)]^{\frac{\omega}{\omega+\varepsilon}} \, d\mu(y) \right\}^{\frac{(\omega+\varepsilon)q}{\omega}} d\mu(z) \\ &\lesssim \fint_{B(x,2A_0C_0\delta^l)} \sum_{m=l+1}^{\infty} \delta^{m(\varepsilon'-s)q} \sum_{j=m-2}^{\infty} \delta^{j(s-\varepsilon')q} \\ &\quad \times \left\{ M\left([g_j\mathbf{1}_{B(x,\delta^{l-2})}]^{\frac{\omega}{\omega+\varepsilon}}\right)(z) \right\}^{\frac{(\omega+\varepsilon)q}{\omega}} d\mu(z) \\ &\lesssim \fint_{B(x,2A_0C_0\delta^l)} \sum_{j=l-1}^{\infty} \left\{ M\left([g_j\mathbf{1}_{B(x,\delta^{l-2})}]^{\frac{\omega}{\omega+\varepsilon}}\right)(z) \right\}^{\frac{(\omega+\varepsilon)q}{\omega}} d\mu(z) \\ &\lesssim \sum_{j=l-2}^{\infty} \fint_{B(x,\delta^{l-2})} [g_j(z)]^q \, d\mu(z) \lesssim \|u\|^q_{\dot{M}^s_{\infty,q}(\mathcal{X})}. \end{aligned}$$

Combining the estimates of both $\mathrm{Y}_3$ and $\mathrm{Y}_4$, we conclude that, for any $l \in \mathbb{Z}$ and $x \in \mathcal{X}$,

$$\fint_{B(x,2A_0C_0\delta^l)} \sum_{k=l}^{\infty} \delta^{-ksq} \sup_{\phi \in \mathcal{F}_k(z)} |\langle u, \phi \rangle|^q \, d\mu(z) \lesssim \|u\|^q_{\dot{M}^s_{\infty,q}(\mathcal{X})},$$

which, together with Lemma 2.4(iii), implies the desired estimate in this case.

Thus, we have $\dot{M}^s_{p,q}(\mathcal{X}) \subset \mathcal{A}\dot{F}^s_{p,q}(\mathcal{X})$.

We finally show $\mathcal{A}\dot{F}^s_{p,q}(\mathcal{X}) \subset \dot{M}^s_{p,q}(\mathcal{X})$. Assume that $f \in \mathcal{A}\dot{F}^s_{p,q}(\mathcal{X})$. By Lemma 3.19 and its proof, we conclude that there exist a subsequence $\{Q_{k_j} f\}_{j\in\mathbb{N}}$ and a constant $C \in \mathbb{R}$ such that, for almost every $x \in \mathcal{X}$, $\lim_{j\to\infty} Q_{k_j} f(x) = f(x) - C$. For any $k \in \mathbb{Z}$ and $x \in \mathcal{X}$, let

$$g_k(x) := \delta^{-ks} \sup_{\phi\in\mathcal{F}_k(x)} |\langle f, \phi\rangle|.$$

For almost all $x, y \in \mathcal{X}$ and $k_0 \in \mathbb{Z}$ satisfying $\delta^{k_0+1} \le d(x, y) < \delta^{k_0}$, we find a $j_0 \in \mathbb{N}$ satisfying $k_0 \le k_{j_0}$ and we then estimate

$$\begin{aligned}
|f(x) - f(y)| &= |f(x) - Q_{k_{j_0}} f(x) + Q_{k_{j_0}} f(x) - Q_{k_{j_0}} f(y) + Q_{k_{j_0}} f(y) - f(y)| \\
&\le |Q_{k_{j_0}} f(x) - Q_{k_{j_0}} f(y)| + \sum_{j=j_0}^{\infty} \Big[|Q_{k_{j+1}} f(x) - Q_{k_j} f(x)| \\
&\quad + |Q_{k_{j+1}} f(x) - Q_{k_j} f(x)|\Big] \\
&\le \delta^{k_{j_0} s}[g_{k_{j_0}}(x) + g_{k_{j_0}}(y)] + \sum_{k=k_0}^{\infty} \Big[|Q_{k+1} f(x) - Q_k f(x)| \\
&\quad + |Q_{k+1} f(x) - Q_k f(x)|\Big] \\
&\le 2 \sum_{k=k_0}^{\infty} \delta^{ks}[g_k(x) + g_k(y)].
\end{aligned}$$

For any $k \in \mathbb{Z}$, define

$$h_k := 2 \sum_{j=k}^{\infty} \delta^{(-k+j-1)s} g_j.$$

We then have, for almost all $x, y \in \mathcal{X}$ with $\delta^{k+1} \le d(x, y) < \delta^k$,

$$\begin{aligned}
|f(x) - f(y)| &\le 2 \sum_{j=k}^{\infty} \delta^{js}[g_j(x) + g_j(y)] \le \delta^{(k+1)s}[h_k(x) + h_k(y)] \\
&\le [d(x, y)]^s[h_k(x) + h_k(y)],
\end{aligned}$$

which implies that $\{h_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^s(f)$. Note that, by the Hölder inequality when $q \in (1, \infty]$, or (3.1) when $q \in (\omega/(\omega + s), 1]$, we obtain

$$\sum_{k=-\infty}^{\infty} h_k^q \lesssim \sum_{k=-\infty}^{\infty} \Big[\sum_{j=k}^{\infty} \delta^{(-k+j-1)s} g_j\Big]^q \lesssim \sum_{k=-\infty}^{\infty} \sum_{j=k}^{\infty} \delta^{(-k+j-1)sq/2} g_j^q \lesssim \sum_{j=-\infty}^{\infty} g_j^q,$$

which implies that, for any given $p \in (\omega/(\omega + s), \infty)$ and $q \in (\omega/(\omega + s), \infty]$,

$$\|f\|_{\dot{M}^s_{p,q}(\mathcal{X})} \le \Big\| \Big(\sum_{k=\infty}^{\infty} h_k^q\Big)^{1/q} \Big\|_{L^p(\mathcal{X})} \lesssim \Big\| \Big(\sum_{j=\infty}^{\infty} g_j^q\Big)^{1/q} \Big\|_{L^p(\mathcal{X})} \sim \|f\|_{\mathcal{A}\dot{F}^s_{p,q}(\mathcal{X})}.$$

When $p=\infty$, by the Hölder inequality when $q\in(1,\infty]$, or (3.1) when $q\in(\omega/(\omega+s),1)$, we have, for any $l\in\mathbb{Z}$,

$$\sum_{k=l}^{\infty} h_k^q \lesssim \sum_{k=l}^{\infty}\Big[\sum_{j=k}^{\infty}\delta^{(-k+j-1)s}g_j\Big]^q \lesssim \sum_{k=l}^{\infty}\sum_{j=k}^{\infty}\delta^{(-k+j-1)sq/2}g_j^q \lesssim \sum_{j=l}^{\infty} g_j^q,$$

which implies that, for any $x\in\mathcal{X}$,

$$\sum_{k=l}^{\infty} \fint_{B(x,\delta^l)} [h_k(y)]^q\,d\mu(y) \lesssim \sum_{j=l}^{\infty} \fint_{B(x,\delta^l)} [g_j(y)]^q\,d\mu(y).$$

From this, we deduce that $\|f\|_{\dot{M}^s_{\infty,q}(\mathcal{X})} \lesssim \|f\|_{\mathcal{A}\dot{F}^s_{\infty,q}(\mathcal{X})}$, which completes the proof of (i), and hence of Theorem 3.10. ∎

*Proof of Theorem 2.16.* The theorem is a direct corollary of both Theorems 3.3 and 3.10; we omit the details. ∎

**4. Pointwise characterization of inhomogeneous Besov and Triebel–Lizorkin spaces.** In this section, we establish the inhomogeneous version of Theorem 2.16. Let us begin with the concept of inhomogeneous approximations of the identity with exponential decay (see [24, Definition 6.1]).

Definition 4.1. Let $\eta\in(0,1)$ be as in Definition 2.5. A sequence $\{Q_k\}_{k\in\mathbb{Z}_+}$ of bounded linear integral operators on $L^2(\mathcal{X})$ is called an *inhomogeneous approximation of the identity with exponential decay* (for short, exp-IATI) if $\{Q_k\}_{k\in\mathbb{Z}_+}$ has the following properties:

(i) $\sum_{k=0}^{\infty} Q_k = I$ in $L^2(\mathcal{X})$;
(ii) for any $k\in\mathbb{N}$, $Q_k$ satisfies (ii) through (v) of Definition 2.5;
(iii) $Q_0$ satisfies (ii), (iii), and (iv) of Definition 2.5 with $k=0$ but without the decay factor

$$\exp\{-\nu[\max\{d(x,\mathcal{Y}^0),d(y,\mathcal{Y}^0)\}]^a\};$$

moreover, for any $x\in\mathcal{X}$,

$$\int_{\mathcal{X}} Q_0(x,y)\,d\mu(y) = 1 = \int_{\mathcal{X}} Q_0(y,x)\,d\mu(y). \tag{4.1}$$

Remark 4.2. As was pointed out in [24, Remark 6.2], the existence of an exp-IATI on $\mathcal{X}$ is guaranteed by the main results from [4]. In Definition 4.1, due to (4.1), we do not need $\operatorname{diam}\mathcal{X}=\infty$ to guarantee the existence of an exp-IATI on $\mathcal{X}$. In other words, $\operatorname{diam}\mathcal{X}$ can be finite or infinite.

Based on the concept of exp-IATIs, He et al. established the following inhomogeneous discrete Calderón reproducing formulae in [24, Theorems 6.10 and 6.13].

LEMMA 4.3. *Let* $\{Q_k\}_{k\in\mathbb{Z}_+}$ *be an* exp-IATI *and* $\beta,\gamma\in(0,\eta)$ *with* $\eta$ *as in Definition* 2.5. *For any* $k\in\mathbb{Z}_+$, $\alpha\in\mathcal{A}_k$, *and* $m\in\{1,\dots,N(k,\alpha)\}$, *suppose that* $y_\alpha^{k,m}$ *is an arbitrary point in* $Q_\alpha^{k,m}$. *Then there exist an* $N\in\mathbb{N}$ *and a sequence* $\{\widetilde{Q}_k\}_{k\in\mathbb{Z}_+}$ *of bounded linear integral operators on* $L^2(\mathcal{X})$ *such that, for any* $f\in(\mathcal{G}_0^\eta(\beta,\gamma))'$,

$$\begin{aligned} f(\cdot) &= \sum_{\alpha\in\mathcal{A}_0}\sum_{m=1}^{N(0,\alpha)}\int_{Q_\alpha^{0,m}}\widetilde{Q}_0(\cdot,y)\,d\mu(y)Q_{\alpha,1}^{0,m}(f)\\ &\quad+\sum_{k=1}^{N}\sum_{\alpha\in\mathcal{A}_k}\sum_{m=1}^{N(k,\alpha)}\mu(Q_\alpha^{k,m})\widetilde{Q}_k(\cdot,y_\alpha^{k,m})Q_{\alpha,1}^{k,m}(f)\\ &\quad+\sum_{k=N+1}^{\infty}\sum_{\alpha\in\mathcal{A}_k}\sum_{m=1}^{N(k,\alpha)}\mu(Q_\alpha^{k,m})\widetilde{Q}_k(\cdot,y_\alpha^{k,m})Q_kf(y_\alpha^{k,m}) \end{aligned}$$

*in* $(\mathcal{G}_0^\eta(\beta,\gamma))'$, *where, for any* $k\in\{0,\dots,N\}$, $\alpha\in\mathcal{A}_k$, $m\in\{1,\dots,N(k,\alpha)\}$, *and* $x\in\mathcal{X}$,

$$Q_{\alpha,1}^{k,m}(x):=\frac{1}{\mu(Q_\alpha^{k,m})}\int_{Q_\alpha^{k,m}}Q_k(y,x)\,d\mu(y),$$

*and* $Q_{\alpha,1}^{k,m}(f):=\langle f,Q_{\alpha,1}^{k,m}\rangle$. *Moreover, for any* $k\in\mathbb{Z}_+$, *the kernel of* $\widetilde{Q}_k$, *still denoted by* $\widetilde{Q}_k$, *satisfies* (3.2), (3.3), *and the following integral condition: for any* $x\in\mathcal{X}$,

$$\int_{\mathcal{X}}\widetilde{Q}_k(x,y)\,d\mu(y)=\int_{\mathcal{X}}\widetilde{Q}_k(y,x)\,d\mu(y)=\begin{cases}1 & \text{if } k\in\{0,\dots,N\},\\ 0 & \text{if } k\in\{N+1,N+2,\dots\}.\end{cases}$$

We now recall the concepts of inhomogeneous spaces $B_{p,q}^s(\mathcal{X})$ and $F_{p,q}^s(\mathcal{X})$ introduced in [48]. To this end, for any dyadic cube $Q$ and any non-negative measurable function $f$ on $\mathcal{X}$, let

$$m_Q(f):=\frac{1}{\mu(Q)}\int_Q f(y)\,d\mu(y).$$

DEFINITION 4.4. Let $\beta,\gamma\in(0,\eta)$ with $\eta$ as in Definition 2.5, and $s\in(-(\beta\wedge\gamma),\beta\wedge\gamma)$. Let $\{Q_k\}_{k\in\mathbb{Z}_+}$ be an exp-IATI and $N\in\mathbb{N}$ as in Lemma 4.3.

(i) If $p\in(p(s,\beta\wedge\gamma),\infty]$ with $p(s,\beta\wedge\gamma)$ as in (1.1), and $q\in(0,\infty]$, then the *inhomogeneous Besov space* $B_{p,q}^s(\mathcal{X})$ is defined to be the set of all the $f \in (\mathcal{G}_0^\eta(\beta,\gamma))'$ such that

$$\|f\|_{B^s_{p,q}(\mathcal{X})} := \Big\{\sum_{k=0}^{N}\sum_{\alpha\in\mathcal{A}_k}\sum_{m=1}^{N(k,\alpha)} \mu(Q_\alpha^{k,m})[m_{Q_\alpha^{k,m}}(|Q_k f|)]^p\Big\}^{1/p} + \Big[\sum_{k=N+1}^{\infty} \delta^{-ksq}\|Q_k f\|^q_{L^p(\mathcal{X})}\Big]^{1/q} < \infty$$

with the usual modifications when $p=\infty$ or $q=\infty$.

(ii) If $p \in (p(s,\beta\wedge\gamma),\infty)$ and $q \in (p(s,\beta\wedge\gamma),\infty]$, then the *inhomogeneous Triebel–Lizorkin space* $F^s_{p,q}(\mathcal{X})$ is defined to be the set of all the $f \in (\mathcal{G}_0^\eta(\beta,\gamma))'$ such that

$$\|f\|_{F^s_{p,q}(\mathcal{X})} := \Big\{\sum_{k=0}^{N}\sum_{\alpha\in\mathcal{A}_k}\sum_{m=1}^{N(k,\alpha)} \mu(Q_\alpha^{k,m})[m_{Q_\alpha^{k,m}}(|Q_k f|)]^p\Big\}^{1/p} + \Big\|\Big(\sum_{k=N+1}^{\infty} \delta^{-ksq}|Q_k f|^q\Big)^{1/q}\Big\|_{L^p(\mathcal{X})} < \infty$$

with the usual modification when $q=\infty$.

Remark 4.5. (i) We point out that we *do not* need the assumption $\mu(\mathcal{X})=\infty$ in Definitions 4.1 and 4.4.

(ii) It was proved in [48, Propositions 4.3 and 4.4] that, when $\beta,\gamma,s,p$, and $q$ are as in Definition 4.4, the inhomogeneous Besov and Triebel–Lizorkin spaces are independent of the choices of both the exp-IATIs and the spaces of distributions.

We next recall the concepts of 1-exp-IATIs (see, for instance, [26, Definition 3.1]) and the local Hardy space $h^p(\mathcal{X})$ (see, for instance, [26, Section 3]).

Definition 4.6. Let $\eta \in (0,1)$ be as in Definition 2.5. A sequence $\{P_k\}_{k\in\mathbb{Z}_+}$ of bounded linear integral operators on $L^2(\mathcal{X})$ is called an *inhomogeneous approximation of the identity with exponential decay and integration* 1 (for short, 1-exp-IATI) if $\{P_k\}_{k\in\mathbb{Z}_+}$ has the following properties:

(i) for any $k\in\mathbb{Z}_+$, $P_k$ satisfies both (ii) and (iii) of Definition 2.5 but without the term

$$\exp\{-\nu[\max\{d(x,\mathcal{Y}^k),d(y,\mathcal{Y}^k)\}]^a\};$$

(ii) for any $k\in\mathbb{Z}_+$ and $x\in\mathcal{X}$,

$$\int_{\mathcal{X}} P_k(x,y)\,d\mu(y) = 1 = \int_{\mathcal{X}} P_k(y,x)\,d\mu(y);$$

(iii) if $Q_0 := P_0$ and $Q_k := P_k - P_{k-1}$ for any $k \in \mathbb{N}$, then $\{Q_k\}_{k\in\mathbb{Z}_+}$ is an exp-IATI.

DEFINITION 4.7. Let $\mathcal{X}$ be a space of homogeneous type. Let $\{P_k\}_{k\in\mathbb{Z}}$ be a 1-exp-IATI. The *local radial maximal function* $\mathcal{M}_0^+(f)$ of $f$ is defined by setting, for any $x \in \mathcal{X}$,

$$\mathcal{M}_0^+(f)(x) := \max\Big\{\max_{k\in\{0,\dots,N\}}\Big\{\sum_{\alpha\in\mathcal{A}_k}\sum_{m=1}^{N(k,\alpha)}\sup_{z\in Q_\alpha^{k,m}}|P_kf(z)|\mathbf{1}_{Q_\alpha^{k,m}}(x)\Big\},\ \sup_{k\in\{N+1,N+2,\dots\}}|P_kf(x)|\Big\},$$

where $N \in \mathbb{N}$ is as in Lemma 4.3. For any $p \in (0,\infty)$, the *local Hardy space* $h^p(\mathcal{X})$ is defined by setting

$$h^p(\mathcal{X}) := \{f \in (\mathcal{G}_0^\eta(\beta,\gamma))' : \|f\|_{h^p(\mathcal{X})} := \|\mathcal{M}_0^+(f)\|_{L^p(\mathcal{X})} < \infty\}.$$

REMARK 4.8. In [26, Theorem 3.3], it was shown that, when $p \in (1,\infty]$, then $h^p(\mathcal{X}) = L^p(\mathcal{X})$. Also, in [48, Theorem 6.13], it was proved that, when $p \in (1,\infty)$, $F^0_{p,2} = L^p(\mathcal{X})$. Moreover, the Littlewood–Paley $g$-function characterization of $h^p(\mathcal{X})$ in [26, Theorem 5.7] implies that, for any given $p \in (\omega/(\omega+\eta), 1]$, $F^0_{p,2}(\mathcal{X}) = h^p(\mathcal{X})$.

Now, we introduce the concepts of inhomogeneous Hajłasz–Sobolev spaces, Hajłasz–Triebel–Lizorkin spaces, and Hajłasz–Besov spaces.

DEFINITION 4.9. Let $s \in (0,\infty)$.

(i) Let $p \in (0,\infty)$. The *inhomogeneous Hajłasz–Sobolev space* $M^{s,p}(\mathcal{X})$ is defined to be the set of all the measurable functions $u$ on $\mathcal{X}$ such that

$$\|u\|_{M^{s,p}(\mathcal{X})} := \|u\|_{h^p(\mathcal{X})} + \|u\|_{\dot{M}^{s,p}(\mathcal{X})} < \infty.$$

(ii) Let $p,q \in (0,\infty]$. The *inhomogeneous Hajłasz–Triebel–Lizorkin space* $M^s_{p,q}(\mathcal{X})$ is defined to be the set of all the measurable functions $u$ on $\mathcal{X}$ such that

$$\|u\|_{\dot{M}^s_{p,q}(\mathcal{X})} := \|u\|_{h^p(\mathcal{X})} + \|u\|_{\dot{M}^s_{p,q}(\mathcal{X})} < \infty.$$

(iii) Let $p,q \in (0,\infty]$. The *inhomogeneous Hajłasz–Besov space* $N^s_{p,q}(\mathcal{X})$ is defined to be the set of all the measurable functions $u$ on $\mathcal{X}$ such that

$$\|u\|_{N^s_{p,q}(\mathcal{X})} := \|u\|_{h^p(\mathcal{X})} + \|u\|_{\dot{N}^s_{p,q}(\mathcal{X})} < \infty.$$

The following statement is the inhomogeneous version of Theorem 2.16.

THEOREM 4.10. *Let $\omega$ and $\eta$ be, respectively, as in* (2.2) *and Definition* 2.5*, $\beta,\gamma \in (0,\eta)$, $s \in (0,\beta\wedge\gamma)$, and $p,q$ be as in Definition* 4.4*. Assume that $\mu(\mathcal{X}) = \infty$ and the measure $\mu$ of $\mathcal{X}$ has a weak lower bound $Q = \omega$.*

(i) *If* $p\in(\omega/(\omega+s),\infty)$ *and* $q\in(\omega/(\omega+s),\infty]$, *then* $M^s_{p,q}(\mathcal{X})=F^s_{p,q}(\mathcal{X})$.
(ii) *If* $p\in(\omega/(\omega+s),\infty]$ *and* $q\in(0,\infty]$, *then* $N^s_{p,q}(\mathcal{X})=B^s_{p,q}(\mathcal{X})$.

To prove Theorem 4.10, we first establish a relation between local Hardy spaces and inhomogeneous Besov and Triebel–Lizorkin spaces, whose RD-space version was obtained in [60, Theorem 1.2]. We point out that the proof of [60, Theorem 1.2] just depends on the size, the regularity, and the cancellation conditions of the approximation of the identity (for short, ATI), but *does not* involve the reverse doubling condition of the underlying space and the bounded support of ATI. As a result, the proof of [60, Theorem 1.2] is still valid in any space of homogeneous type due to Definition 2.5(v) and Lemma 2.6; we omit the details.

THEOREM 4.11. *Let* $\beta,\gamma\in(0,\eta)$ *with* $\eta$ *as in Definition* 2.5, $s\in(0,\beta\wedge\gamma)$, *and* $p\in(\omega/(\omega+s),\infty)$ *with* $\omega$ *as in* (2.2). *Assume* $\mu(\mathcal{X})=\infty$. *Let* $\{Q_k\}_{k\in\mathbb{Z}}$ *be an* exp-ATI.

(i) *If* $q\in(0,\infty]$, *then* $f\in B^s_{p,q}(\mathcal{X})$ *if and only if* $f\in h^p(\mathcal{X})$ *and*

$$\mathrm{J}_1:=\Big[\sum_{k=-\infty}^{\infty}\delta^{-ksq}\|Q_kf\|^q_{L^p(X)}\Big]^{1/q}<\infty.$$

*Moreover,* $\|f\|_{B^s_{p,q}(\mathcal{X})}$ *is equivalent to* $\|f\|_{h^p(\mathcal{X})}+\mathrm{J}_1$ *with positive equivalence constants independent of* $f$.

(ii) *If* $q\in(\omega/(\omega+s),\infty]$, *then* $f\in F^s_{p,q}(\mathcal{X})$ *if and only if* $f\in h^p(\mathcal{X})$ *and*

$$\mathrm{J}_2:=\Big\|\Big(\sum_{k=-\infty}^{\infty}\delta^{-ksq}|Q_kf|^q\Big)^{1/q}\Big\|_{L^p(X)}<\infty.$$

*Moreover,* $\|f\|_{F^s_{p,q}(\mathcal{X})}$ *is equivalent to* $\|f\|_{h^p(\mathcal{X})}+\mathrm{J}_2$ *with positive equivalence constants independent of* $f$.

REMARK 4.12. Usually, it makes no sense to write the conclusions of Theorem 4.11 as $B^s_{p,q}(\mathcal{X})=h^p(\mathcal{X})\cap\dot{B}^s_{p,q}(\mathcal{X})$ and $F^s_{p,q}(\mathcal{X})=h^p(\mathcal{X})\cap\dot{F}^s_{p,q}(\mathcal{X})$ because homogeneous and inhomogeneous spaces are defined via different kinds of spaces of distributions (see [60, Remark 1.1(iv)]).

*Proof of Theorem 4.10.* We only prove (i) because the proof of (ii) is similar. We first show $M^s_{p,q}(\mathcal{X})\subset F^s_{p,q}(\mathcal{X})$. To this end, assume $u\in M^s_{p,q}(\mathcal{X})$. By Definition 4.9, we know that $u\in h^p(\mathcal{X})$ and $u\in\dot{M}^s_{p,q}(\mathcal{X})$. We then consider two cases of $p$.

CASE 1: $p\in(1,\infty)$. In this case, since $p\in(1,\infty)$, from [26, Theorem 3.3], it follows that $u\in L^p(\mathcal{X})$. Moreover, by Theorem 2.16, we find that $u\in\dot{F}^s_{p,q}(\mathcal{X})$. These, together with [48, Theorem 6.12], imply that $u\in F^s_{p,q}(\mathcal{X})$ and $\|u\|_{F^s_{p,q}(\mathcal{X})}\lesssim\|u\|_{M^s_{p,q}(\mathcal{X})}$.

CASE 2: $p \in (\omega/(\omega+s), 1]$. In this case, by Theorem 2.16, we know that $u \in \dot{F}^s_{p,q}(\mathcal{X})$ and $\|u\|_{\dot{F}^s_{p,q}(\mathcal{X})} \lesssim \|u\|_{\dot{M}^s_{p,q}(\mathcal{X})}$. Then, using Theorem 4.11(ii), we conclude that $u \in F^s_{p,q}(\mathcal{X})$ and $\|u\|_{F^s_{p,q}(\mathcal{X})} \lesssim \|u\|_{M^s_{p,q}(\mathcal{X})}$.

Next, we show $F^s_{p,q}(\mathcal{X}) \subset M^s_{p,q}(\mathcal{X})$. To this end, assume that $u \in F^s_{p,q}(\mathcal{X})$. As above, we consider two cases of $p$.

CASE 1: $p \in (1, \infty)$. In this case, from [48, Theorem 6.12], it follows that $u \in L^p(\mathcal{X}) \cap \dot{F}^s_{p,q}(\mathcal{X})$ and

$$\|u\|_{F^s_{p,q}(\mathcal{X})} \sim \|u\|_{L^p(\mathcal{X})} + \|u\|_{\dot{F}^s_{p,q}(\mathcal{X})}.$$

By [26, Theorem 3.3] again, we conclude that $u \in h^p(\mathcal{X})$ and $\|u\|_{h^p(\mathcal{X})} \sim \|u\|_{L^p(\mathcal{X})}$. Moreover, from Theorem 2.16, we infer that $u \in \dot{M}^s_{p,q}(\mathcal{X})$ and $\|u\|_{\dot{M}^s_{p,q}(\mathcal{X})} \sim \|u\|_{\dot{F}^s_{p,q}(\mathcal{X})}$, which further implies that $u \in M^s_{p,q}(\mathcal{X})$ and

$$\|u\|_{M^s_{p,q}(\mathcal{X})} \lesssim \|u\|_{F^s_{p,q}(\mathcal{X})}.$$

CASE 2: $p \in (\omega/(\omega+s), 1]$. In this case, since $u \in F^s_{p,q}(\mathcal{X})$, from [48, Proposition 4.4], it follows that $u \in (\mathcal{G}^\eta_0(\beta,\gamma))'$ with $\beta$ and $\gamma$ as in Definition 4.4. As $p \in (\omega/(\omega+s), 1]$, we know that $s > \omega(1/p-1)$. Choosing a $\beta_0 \in (0,\eta)$ and a $\gamma_0 \in (s,\eta) \subset (\omega[1/p-1], \eta)$, we then find a $u \in (\mathcal{G}^\eta_0(\beta_0,\gamma_0))' \subset (\mathring{\mathcal{G}}^\eta_0(\beta_0,\gamma_0))'$. From this, Theorem 4.11(ii), and [48, Proposition 3.15], we deduce that $u \in \dot{F}^s_{p,q}(\mathcal{X})$ and

$$\|u\|_{F^s_{p,q}(\mathcal{X})} \sim \|u\|_{L^p(\mathcal{X})} + \|u\|_{\dot{F}^s_{p,q}(\mathcal{X})},$$

which implies that $u \in M^s_{p,q}(\mathcal{X})$ and $\|u\|_{M^s_{p,q}(\mathcal{X})} \lesssim \|u\|_{F^s_{p,q}(\mathcal{X})}$.

This finishes the proof of (i), and hence of Theorem 4.10. ∎

REMARK 4.13. We point out that it is not clear whether or not Theorem 4.10 still holds true when $\mu(\mathcal{X}) \neq \infty$, because the existence of exp-ATIs in Theorem 4.11 needs $\mu(\mathcal{X}) = \infty$ [see Remark 2.9(i)].

**Acknowledgements.** The second author would like to thank Ziyi He for his valuable discussions and suggestions on this article. The authors would also like to thank both referees for their careful reading and several remarks which indeed improved the presentation of this article.

This project is partially supported by the National Key Research and Development Program of China (Grant No. 2020YFA0712900) and the National Natural Science Foundation of China (Grant Nos. 11971058, 12071197 and 12122102).

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