

1

FUNCTIONS AND LIMITS

Calculus is fundamentally different from the mathematics that you have studied previously. Calculus is less static and more dynamic. It is concerned with change and motion; it deals with quantities that approach other quantities. So in this first chapter we begin our study of calculus by investigating how the values of functions change and approach limits.

1.1

FUNCTIONS AND THEIR REPRESENTATIONS

Functions arise whenever one quantity depends on another. Consider the following four situations.

A. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A , and we say that A is a *function* of r .

B. The human population of the world P depends on the time t . The table gives estimates of the world population $P(t)$ at time t , for certain years. For instance,

$$P(1950) \approx 2,560,000,000$$

But for each value of the time t there is a corresponding value of P , and we say that P is a function of t .

C. The cost C of mailing a first-class letter depends on the weight w of the letter. Although there is no simple formula that connects w and C , the post office has a rule for determining C when w is known.

D. The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t . Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t , the graph provides a corresponding value of a .

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

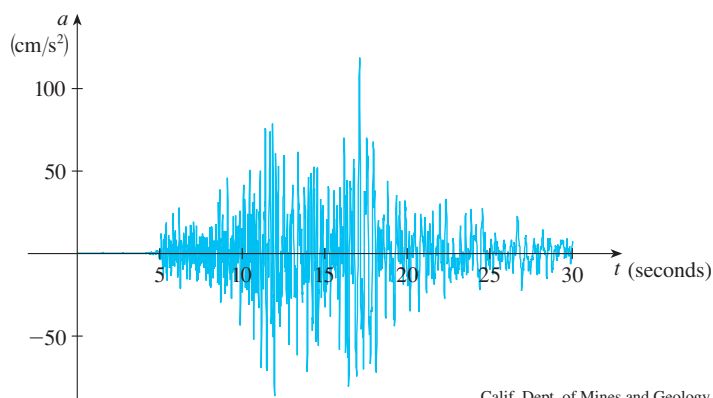


FIGURE 1

Vertical ground acceleration during the Northridge earthquake

Calif. Dept. of Mines and Geology

Each of these examples describes a rule whereby, given a number (r , t , w , or t), another number (A , P , C , or a) is assigned. In each case we say that the second number is a function of the first number.

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B .

We usually consider functions for which the sets A and B are sets of real numbers. The set A is called the **domain** of the function. The number $f(x)$ is the **value of f at x** and is read “ f of x .” The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain. A symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**. A symbol that represents a number in the *range* of f is called a **dependent variable**. In Example A, for instance, r is the independent variable and A is the dependent variable.



FIGURE 2
Machine diagram for a function f

It’s helpful to think of a function as a **machine** (see Figure 2). If x is in the domain of the function f , then when x enters the machine, it’s accepted as an input and the machine produces an output $f(x)$ according to the rule of the function. Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

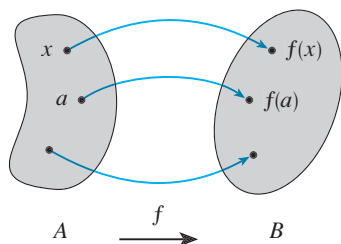


FIGURE 3
Arrow diagram for f

Another way to picture a function is by an **arrow diagram** as in Figure 3. Each arrow connects an element of A to an element of B . The arrow indicates that $f(x)$ is associated with x , $f(a)$ is associated with a , and so on.

The most common method for visualizing a function is its graph. If f is a function with domain A , then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in A\}$$

(Notice that these are input-output pairs.) In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f .

The graph of a function f gives us a useful picture of the behavior or “life history” of a function. Since the y -coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x . (See Figure 4.) The graph of f also allows us to picture the domain of f on the x -axis and its range on the y -axis as in Figure 5.

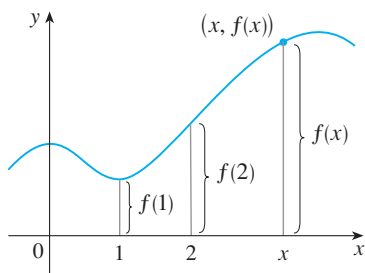


FIGURE 4

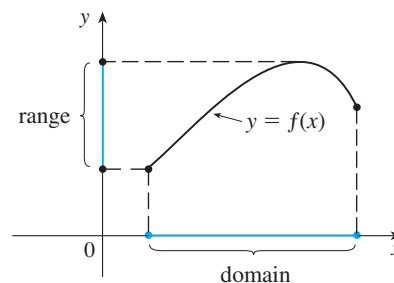


FIGURE 5

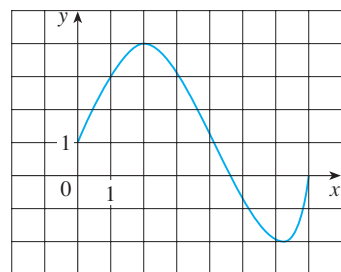


FIGURE 6

EXAMPLE 1 The graph of a function f is shown in Figure 6.

- (a) Find the values of $f(1)$ and $f(5)$.
- (b) What are the domain and range of f ?

SOLUTION

(a) We see from Figure 6 that the point $(1, 3)$ lies on the graph of f , so the value of f at 1 is $f(1) = 3$. (In other words, the point on the graph that lies above $x = 1$ is 3 units above the x -axis.)

When $x = 5$, the graph lies about 0.7 unit below the x -axis, so we estimate that $f(5) \approx -0.7$.

■ The notation for intervals is given on Reference Page 3. The Reference Pages are located at the front and back of the book.

(b) We see that $f(x)$ is defined when $0 \leq x \leq 7$, so the domain of f is the closed interval $[0, 7]$. Notice that f takes on all values from -2 to 4 , so the range of f is

$$\{y \mid -2 \leq y \leq 4\} = [-2, 4]$$

REPRESENTATIONS OF FUNCTIONS

There are four possible ways to represent a function:

- verbally (by a description in words)
- visually (by a graph)
- numerically (by a table of values)
- algebraically (by an explicit formula)

If a single function can be represented in all four ways, it is often useful to go from one representation to another to gain additional insight into the function. But certain functions are described more naturally by one method than by another. With this in mind, let's reexamine the four situations that we considered at the beginning of this section.

A. The most useful representation of the area of a circle as a function of its radius is probably the algebraic formula $A(r) = \pi r^2$, though it is possible to compile a table of values or to sketch a graph (half a parabola). Because a circle has to have a positive radius, the domain is $\{r \mid r > 0\} = (0, \infty)$, and the range is also $(0, \infty)$.

B. We are given a description of the function in words: $P(t)$ is the human population of the world at time t . The table of values of world population provides a convenient representation of this function. If we plot these values, we get the graph (called a *scatter plot*) in Figure 7. It too is a useful representation; the graph allows us to absorb all the data at once. What about a formula? Of course, it's impossible to devise an explicit formula that gives the exact human population $P(t)$ at any time t . But it is possible to find an expression for a function that *approximates* $P(t)$. In fact, we could use a graphing calculator with exponential regression capabilities to obtain the approximation

$$P(t) \approx f(t) = (0.008079266) \cdot (1.013731)^t$$

and Figure 8 shows that it is a reasonably good "fit." The function f is called a *mathematical model* for population growth. In other words, it is a function with an explicit formula that approximates the behavior of our given function. We will see, however, that the ideas of calculus can be applied to a table of values; an explicit formula is not necessary.

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

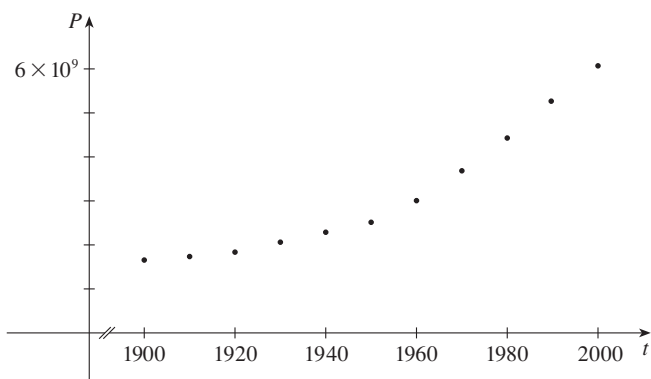


FIGURE 7 Scatter plot of data points for population growth

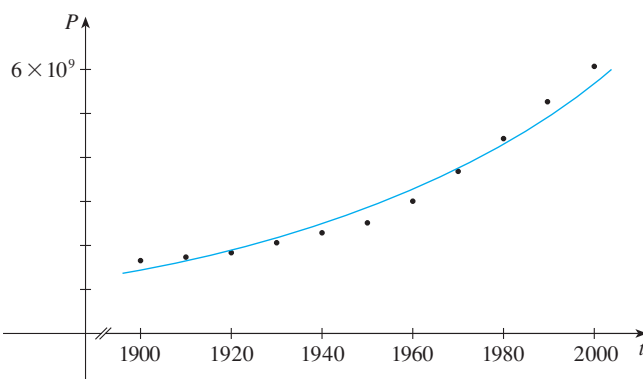


FIGURE 8 Graph of a mathematical model for population growth

■ A function defined by a table of values is called a *tabular* function.

w (ounces)	$C(w)$ (dollars)
$0 < w \leq 1$	0.39
$1 < w \leq 2$	0.63
$2 < w \leq 3$	0.87
$3 < w \leq 4$	1.11
$4 < w \leq 5$	1.35
\vdots	\vdots
\vdots	\vdots
$12 < w \leq 13$	3.27

- The function P is typical of the functions that arise whenever we attempt to apply calculus to the real world. We start with a verbal description of a function. Then we may be able to construct a table of values of the function, perhaps from instrument readings in a scientific experiment. Even though we don't have complete knowledge of the values of the function, we will see throughout the book that it is still possible to perform the operations of calculus on such a function.
- C. Again the function is described in words: $C(w)$ is the cost of mailing a first-class letter with weight w . The rule that the US Postal Service used as of 2006 is as follows: The cost is 39 cents for up to one ounce, plus 24 cents for each successive ounce up to 13 ounces. The table of values shown in the margin is the most convenient representation for this function, though it is possible to sketch a graph (see Example 6).
 - D. The graph shown in Figure 1 is the most natural representation of the vertical acceleration function $a(t)$. It's true that a table of values could be compiled, and it is even possible to devise an approximate formula. But everything a geologist needs to know—amplitudes and patterns—can be seen easily from the graph. (The same is true for the patterns seen in electrocardiograms of heart patients and polygraphs for lie-detection.)

In the next example we sketch the graph of a function that is defined verbally.

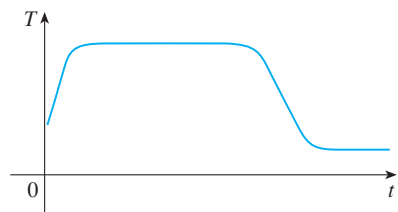


FIGURE 9

EXAMPLE 2 When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

SOLUTION The initial temperature of the running water is close to room temperature because the water has been sitting in the pipes. When the water from the hot-water tank starts flowing from the faucet, T increases quickly. In the next phase, T is constant at the temperature of the heated water in the tank. When the tank is drained, T decreases to the temperature of the water supply. This enables us to make the rough sketch of T as a function of t in Figure 9. ■

■ If a function is given by a formula and the domain is not stated explicitly, the convention is that the domain is the set of all numbers for which the formula makes sense and defines a real number.

EXAMPLE 3 Find the domain of each function.

(a) $f(x) = \sqrt{x + 2}$

(b) $g(x) = \frac{1}{x^2 - x}$

SOLUTION

(a) Because the square root of a negative number is not defined (as a real number), the domain of f consists of all values of x such that $x + 2 \geq 0$. This is equivalent to $x \geq -2$, so the domain is the interval $[-2, \infty)$.

(b) Since

$$g(x) = \frac{1}{x^2 - x} = \frac{1}{x(x - 1)}$$

and division by 0 is not allowed, we see that $g(x)$ is not defined when $x = 0$ or $x = 1$. Thus the domain of g is $\{x \mid x \neq 0, x \neq 1\}$, which could also be written in interval notation as $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$. ■

The graph of a function is a curve in the xy -plane. But the question arises: Which curves in the xy -plane are graphs of functions? This is answered by the following test.

THE VERTICAL LINE TEST A curve in the xy -plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

The reason for the truth of the Vertical Line Test can be seen in Figure 10. If each vertical line $x = a$ intersects a curve only once, at (a, b) , then exactly one functional value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and (a, c) , then the curve can't represent a function because a function can't assign two different values to a .

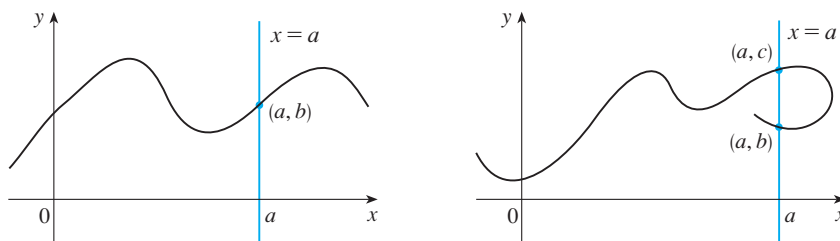


FIGURE 10

PIECEWISE DEFINED FUNCTIONS

The functions in the following three examples are defined by different formulas in different parts of their domains.

EXAMPLE 4 A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

Evaluate $f(0)$, $f(1)$, and $f(2)$ and sketch the graph.

SOLUTION Remember that a function is a rule. For this particular function the rule is the following: First look at the value of the input x . If it happens that $x \leq 1$, then the value of $f(x)$ is $1 - x$. On the other hand, if $x > 1$, then the value of $f(x)$ is x^2 .

$$\text{Since } 0 \leq 1, \text{ we have } f(0) = 1 - 0 = 1.$$

$$\text{Since } 1 \leq 1, \text{ we have } f(1) = 1 - 1 = 0.$$

$$\text{Since } 2 > 1, \text{ we have } f(2) = 2^2 = 4.$$

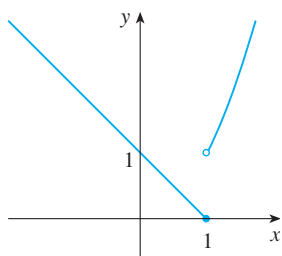


FIGURE 11

How do we draw the graph of f ? We observe that if $x \leq 1$, then $f(x) = 1 - x$, so the part of the graph of f that lies to the left of the vertical line $x = 1$ must coincide with the line $y = 1 - x$, which has slope -1 and y -intercept 1 . If $x > 1$, then $f(x) = x^2$, so the part of the graph of f that lies to the right of the line $x = 1$ must coincide with the graph of $y = x^2$, which is a parabola. This enables us to sketch the graph in Figure 11. The solid dot indicates that the point $(1, 0)$ is included on the graph; the open dot indicates that the point $(1, 1)$ is excluded from the graph. ■

The next example of a piecewise defined function is the absolute value function. Recall that the **absolute value** of a number a , denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0 , so we have

$$|a| \geq 0 \quad \text{for every number } a$$

For example,

$$|3| = 3 \quad |-3| = 3 \quad |0| = 0 \quad |\sqrt{2} - 1| = \sqrt{2} - 1 \quad |3 - \pi| = \pi - 3$$

■ www.stewartcalculus.com

For a more extensive review of absolute values, click on *Review of Algebra*.

In general, we have

$$|a| = a \quad \text{if } a \geq 0$$

$$|a| = -a \quad \text{if } a < 0$$

(Remember that if a is negative, then $-a$ is positive.)

EXAMPLE 5 Sketch the graph of the absolute value function $f(x) = |x|$.

SOLUTION From the preceding discussion we know that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Using the same method as in Example 4, we see that the graph of f coincides with the line $y = x$ to the right of the y -axis and coincides with the line $y = -x$ to the left of the y -axis (see Figure 12).

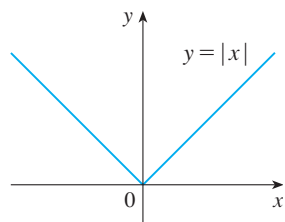


FIGURE 12

EXAMPLE 6 In Example C at the beginning of this section we considered the cost $C(w)$ of mailing a first-class letter with weight w . In effect, this is a piecewise defined function because, from the table of values, we have

$$C(w) = \begin{cases} 0.39 & \text{if } 0 < w \leq 1 \\ 0.63 & \text{if } 1 < w \leq 2 \\ 0.87 & \text{if } 2 < w \leq 3 \\ 1.11 & \text{if } 3 < w \leq 4 \end{cases}$$

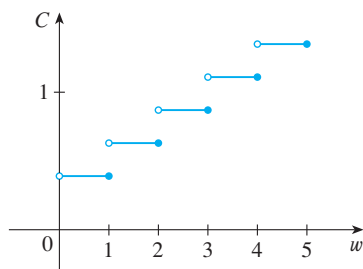


FIGURE 13

The graph is shown in Figure 13. You can see why functions similar to this one are called **step functions**—they jump from one value to the next.

SYMMETRY

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

The geometric significance of an even function is that its graph is symmetric with respect to the y -axis (see Figure 14). This means that if we have plotted the graph of

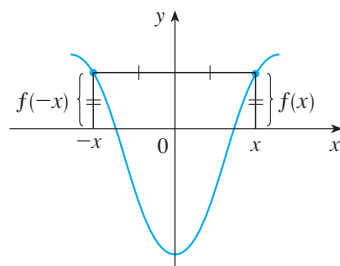


FIGURE 14 An even function

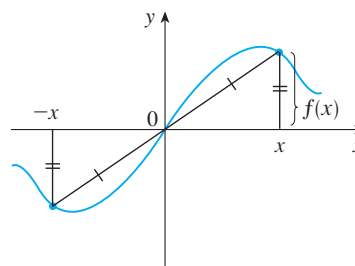
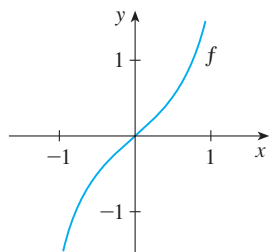
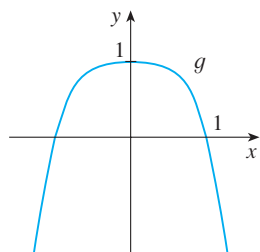


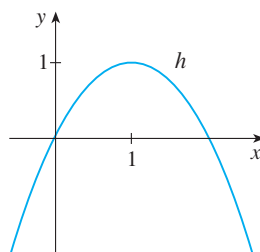
FIGURE 15 An odd function



(a)



(b)



(c)

FIGURE 16

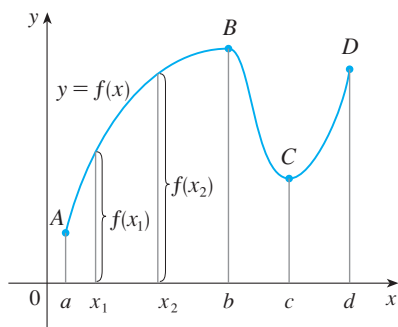


FIGURE 17

f for $x \geq 0$, we obtain the entire graph simply by reflecting this portion about the y -axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

The graph of an odd function is symmetric about the origin (see Figure 15 on page 6). If we already have the graph of f for $x \geq 0$, we can obtain the entire graph by rotating this portion through 180° about the origin.

EXAMPLE 7 Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

SOLUTION

(a)
$$\begin{aligned} f(-x) &= (-x)^5 + (-x) = (-1)^5 x^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore, f is an odd function.

(b)
$$g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

So g is even.

(c)
$$h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd. ■

The graphs of the functions in Example 7 are shown in Figure 16. Notice that the graph of h is symmetric neither about the y -axis nor about the origin.

INCREASING AND DECREASING FUNCTIONS

The graph shown in Figure 17 rises from A to B , falls from B to C , and rises again from C to D . The function f is said to be increasing on the interval $[a, b]$, decreasing on $[b, c]$, and increasing again on $[c, d]$. Notice that if x_1 and x_2 are any two numbers between a and b with $x_1 < x_2$, then $f(x_1) < f(x_2)$. We use this as the defining property of an increasing function.

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

In the definition of an increasing function it is important to realize that the inequality $f(x_1) < f(x_2)$ must be satisfied for *every* pair of numbers x_1 and x_2 in I with $x_1 < x_2$.

You can see from Figure 18 that the function $f(x) = x^2$ is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

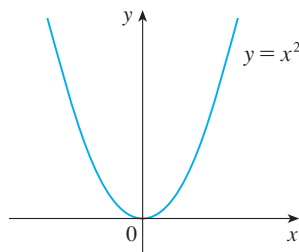
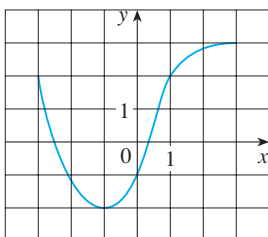


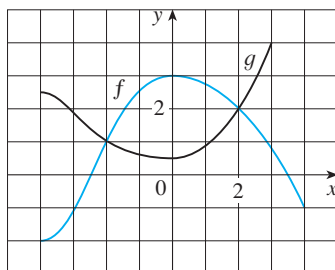
FIGURE 18

1.1 EXERCISES

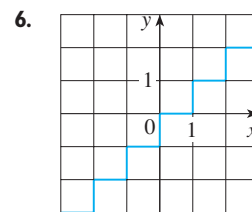
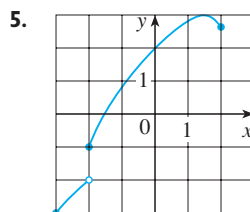
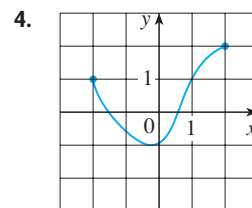
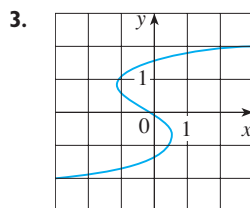
1. The graph of a function f is given.
 - (a) State the value of $f(-1)$.
 - (b) Estimate the value of $f(2)$.
 - (c) For what values of x is $f(x) = 2$?
 - (d) Estimate the values of x such that $f(x) = 0$.
 - (e) State the domain and range of f .
 - (f) On what interval is f increasing?



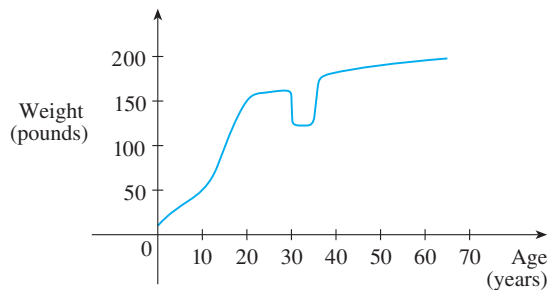
2. The graphs of f and g are given.
 - (a) State the values of $f(-4)$ and $g(3)$.
 - (b) For what values of x is $f(x) = g(x)$?
 - (c) Estimate the solution of the equation $f(x) = -1$.
 - (d) On what interval is f decreasing?
 - (e) State the domain and range of f .
 - (f) State the domain and range of g .



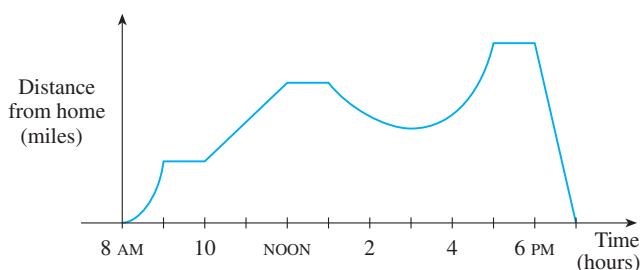
- 3–6 ■ Determine whether the curve is the graph of a function of x . If it is, state the domain and range of the function.



7. The graph shown gives the weight of a certain person as a function of age. Describe in words how this person's weight varies over time. What do you think happened when this person was 30 years old?



8. The graph shown gives a salesman's distance from his home as a function of time on a certain day. Describe in words what the graph indicates about his travels on this day.



9. You put some ice cubes in a glass, fill the glass with cold water, and then let the glass sit on a table. Describe how the temperature of the water changes as time passes. Then sketch a rough graph of the temperature of the water as a function of the elapsed time.
10. Sketch a rough graph of the number of hours of daylight as a function of the time of year.
11. Sketch a rough graph of the outdoor temperature as a function of time during a typical spring day.
12. Sketch a rough graph of the market value of a new car as a function of time for a period of 20 years. Assume the car is well maintained.
13. Sketch the graph of the amount of a particular brand of coffee sold by a store as a function of the price of the coffee.
14. You place a frozen pie in an oven and bake it for an hour. Then you take it out and let it cool before eating it. Describe how the temperature of the pie changes as time passes. Then sketch a rough graph of the temperature of the pie as a function of time.
15. A homeowner mows the lawn every Wednesday afternoon. Sketch a rough graph of the height of the grass as a function of time over the course of a four-week period.
16. A jet takes off from an airport and lands an hour later at another airport, 400 miles away. If t represents the time in minutes since the plane has left the terminal, let $x(t)$ be the horizontal distance traveled and $y(t)$ be the altitude of the plane.
- Sketch a possible graph of $x(t)$.
 - Sketch a possible graph of $y(t)$.
 - Sketch a possible graph of the ground speed.
 - Sketch a possible graph of the vertical velocity.
17. If $f(x) = 3x^2 - x + 2$, find $f(2)$, $f(-2)$, $f(a)$, $f(-a)$, $f(a + 1)$, $2f(a)$, $f(2a)$, $f(a^2)$, $[f(a)]^2$, and $f(a + h)$.
18. A spherical balloon with radius r inches has volume $V(r) = \frac{4}{3}\pi r^3$. Find a function that represents the amount of air required to inflate the balloon from a radius of r inches to a radius of $r + 1$ inches.

- 19–22 ■ Evaluate the difference quotient for the given function. Simplify your answer.

19. $f(x) = 4 + 3x - x^2$, $\frac{f(3+h) - f(3)}{h}$

20. $f(x) = x^3$, $\frac{f(a+h) - f(a)}{h}$

21. $f(x) = \frac{1}{x}$, $\frac{f(x) - f(a)}{x - a}$

22. $f(x) = \frac{x+3}{x+1}$, $\frac{f(x) - f(1)}{x - 1}$

- 23–27 ■ Find the domain of the function.

23. $f(x) = \frac{x}{3x - 1}$

24. $f(x) = \frac{5x + 4}{x^2 + 3x + 2}$

25. $f(t) = \sqrt{t} + \sqrt[3]{t}$

26. $g(u) = \sqrt{u} + \sqrt{4 - u}$

27. $h(x) = \frac{1}{\sqrt[4]{x^2 - 5x}}$

28. Find the domain and range and sketch the graph of the function $h(x) = \sqrt{4 - x^2}$.

- 29–40 ■ Find the domain and sketch the graph of the function.

29. $f(x) = 5$

30. $F(x) = \frac{1}{2}(x + 3)$

31. $f(t) = t^2 - 6t$

32. $H(t) = \frac{4 - t^2}{2 - t}$

33. $g(x) = \sqrt{x - 5}$

34. $F(x) = |2x + 1|$

35. $G(x) = \frac{3x + |x|}{x}$

36. $g(x) = \frac{|x|}{x^2}$

37. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 1 - x & \text{if } x \geq 0 \end{cases}$

38. $f(x) = \begin{cases} 3 - \frac{1}{2}x & \text{if } x \leq 2 \\ 2x - 5 & \text{if } x > 2 \end{cases}$

39. $f(x) = \begin{cases} x + 2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

40. $f(x) = \begin{cases} -1 & \text{if } x \leq -1 \\ 3x + 2 & \text{if } |x| < 1 \\ 7 - 2x & \text{if } x \geq 1 \end{cases}$

41–44 ■ Find an expression for the function whose graph is the given curve.

41. The line segment joining the points $(-2, 1)$ and $(4, -6)$

42. The line segment joining the points $(-3, -2)$ and $(6, 3)$

43. The bottom half of the parabola $x + (y - 1)^2 = 0$

44. The top half of the circle $(x - 1)^2 + y^2 = 1$

45–49 ■ Find a formula for the described function and state its domain.

45. A rectangle has perimeter 20 m. Express the area of the rectangle as a function of the length of one of its sides.

46. A rectangle has area 16 m^2 . Express the perimeter of the rectangle as a function of the length of one of its sides.

47. Express the area of an equilateral triangle as a function of the length of a side.

48. Express the surface area of a cube as a function of its volume.

49. An open rectangular box with volume 2 m^3 has a square base. Express the surface area of the box as a function of the length of a side of the base.

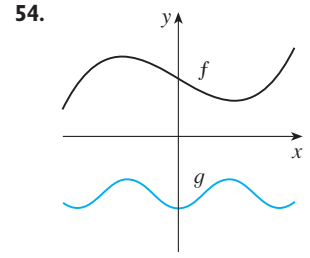
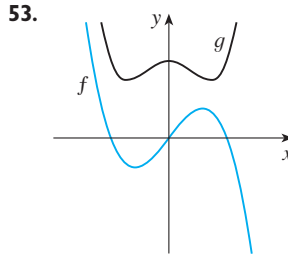
50. A taxi company charges two dollars for the first mile (or part of a mile) and 20 cents for each succeeding tenth of a mile (or part). Express the cost C (in dollars) of a ride as a function of the distance x traveled (in miles) for $0 < x < 2$, and sketch the graph of this function.

51. In a certain country, income tax is assessed as follows. There is no tax on income up to \$10,000. Any income over \$10,000 is taxed at a rate of 10%, up to an income of \$20,000. Any income over \$20,000 is taxed at 15%.
 (a) Sketch the graph of the tax rate R as a function of the income I .
 (b) How much tax is assessed on an income of \$14,000? On \$26,000?
 (c) Sketch the graph of the total assessed tax T as a function of the income I .

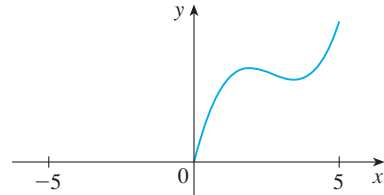
52. The functions in Example 6 and Exercises 50 and 51(a) are called *step functions* because their graphs look like

stairs. Give two other examples of step functions that arise in everyday life.

53–54 ■ Graphs of f and g are shown. Decide whether each function is even, odd, or neither. Explain your reasoning.



55. (a) If the point $(5, 3)$ is on the graph of an even function, what other point must also be on the graph?
 (b) If the point $(5, 3)$ is on the graph of an odd function, what other point must also be on the graph?
56. A function f has domain $[-5, 5]$ and a portion of its graph is shown.
 (a) Complete the graph of f if it is known that f is even.
 (b) Complete the graph of f if it is known that f is odd.



57–62 ■ Determine whether f is even, odd, or neither. If you have a graphing calculator, use it to check your answer visually.

57. $f(x) = \frac{x}{x^2 + 1}$

58. $f(x) = \frac{x^2}{x^4 + 1}$

59. $f(x) = \frac{x}{x + 1}$

60. $f(x) = x|x|$

61. $f(x) = 1 + 3x^2 - x^4$

62. $f(x) = 1 + 3x^3 - x^5$

1.2 A CATALOG OF ESSENTIAL FUNCTIONS

In solving calculus problems you will find that it is helpful to be familiar with the graphs of some commonly occurring functions. These same basic functions are often used to model real-world phenomena, so we begin with a discussion of mathematical modeling. We also review briefly how to transform these functions by shifting, stretching, and reflecting their graphs as well as how to combine pairs of functions by the standard arithmetic operations and by composition.

MATHEMATICAL MODELING

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as the size of a population, the demand for a product, the speed of a falling object, the concentration of a product in a chemical reaction, the life expectancy of a person at birth, or the cost of emission reductions. The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

Figure 1 illustrates the process of mathematical modeling. Given a real-world problem, our first task is to formulate a mathematical model by identifying and naming the independent and dependent variables and making assumptions that simplify the phenomenon enough to make it mathematically tractable. We use our knowledge of the physical situation and our mathematical skills to obtain equations that relate the variables. In situations where there is no physical law to guide us, we may need to collect data (either from a library or the Internet or by conducting our own experiments) and examine the data in the form of a table in order to discern patterns. From this numerical representation of a function we may wish to obtain a graphical representation by plotting the data. The graph might even suggest a suitable algebraic formula in some cases.

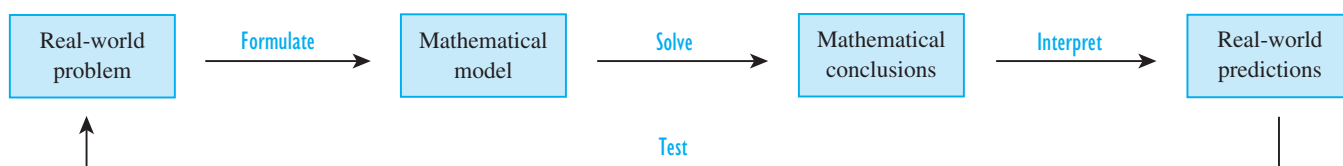


FIGURE 1 The modeling process

The second stage is to apply the mathematics that we know (such as the calculus that will be developed throughout this book) to the mathematical model that we have formulated in order to derive mathematical conclusions. Then, in the third stage, we take those mathematical conclusions and interpret them as information about the original real-world phenomenon by way of offering explanations or making predictions. The final step is to test our predictions by checking against new real data. If the predictions don't compare well with reality, we need to refine our model or to formulate a new model and start the cycle again.

A mathematical model is never a completely accurate representation of a physical situation—it is an *idealization*. A good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions. It is important to realize the limitations of the model. In the end, Mother Nature has the final say.

There are many different types of functions that can be used to model relationships observed in the real world. In what follows, we discuss the behavior and graphs of these functions and give examples of situations appropriately modeled by such functions.

□ LINEAR MODELS

When we say that y is a **linear function** of x , we mean that the graph of the function is a line, so we can use the slope-intercept form of the equation of a line to write a formula for the function as

$$y = f(x) = mx + b$$

where m is the slope of the line and b is the y -intercept.

■ www.stewartcalculus.com

To review the coordinate geometry of lines, click on *Review of Analytic Geometry*.

A characteristic feature of linear functions is that they grow at a constant rate. For instance, Figure 2 shows a graph of the linear function $f(x) = 3x - 2$ and a table of sample values. Notice that whenever x increases by 0.1, the value of $f(x)$ increases by 0.3. So $f(x)$ increases three times as fast as x . Thus the slope of the graph $y = 3x - 2$, namely 3, can be interpreted as the rate of change of y with respect to x .

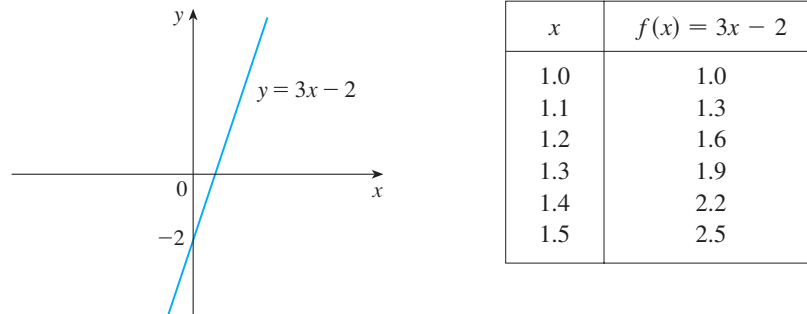


FIGURE 2

EXAMPLE 1

- (a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C , express the temperature T (in $^\circ\text{C}$) as a function of the height h (in kilometers), assuming that a linear model is appropriate.
- (b) Draw the graph of the function in part (a). What does the slope represent?
- (c) What is the temperature at a height of 2.5 km?

SOLUTION

- (a) Because we are assuming that T is a linear function of h , we can write

$$T = mh + b$$

We are given that $T = 20$ when $h = 0$, so

$$20 = m \cdot 0 + b = b$$

In other words, the y -intercept is $b = 20$.

We are also given that $T = 10$ when $h = 1$, so

$$10 = m \cdot 1 + 20$$

The slope of the line is therefore $m = 10 - 20 = -10$ and the required linear function is

$$T = -10h + 20$$

- (b) The graph is sketched in Figure 3. The slope is $m = -10^\circ\text{C}/\text{km}$, and this represents the rate of change of temperature with respect to height.

- (c) At a height of $h = 2.5$ km, the temperature is

$$T = -10(2.5) + 20 = -5^\circ\text{C}$$

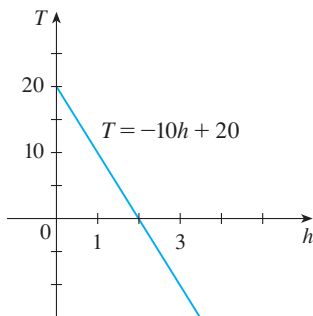


FIGURE 3

□ POLYNOMIALS

A function P is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \dots, a_n$ are constants called the **coefficients** of the polynomial. The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the **degree** of the polynomial is n . For example, the function

$$P(x) = 2x^6 - x^4 + \frac{2}{3}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a linear function. A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**. Its graph is always a parabola obtained by shifting the parabola $y = ax^2$. The parabola opens upward if $a > 0$ and downward if $a < 0$. (See Figure 4.)

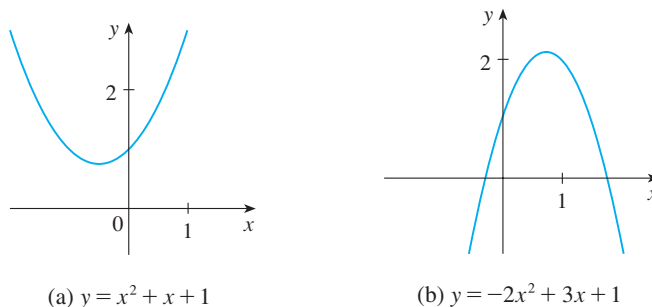


FIGURE 4
The graphs of quadratic functions are parabolas.

A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d \quad a \neq 0$$

and is called a **cubic function**. Figure 5 shows the graph of a cubic function in part (a) and graphs of polynomials of degrees 4 and 5 in parts (b) and (c). We will see later why the graphs have these shapes.

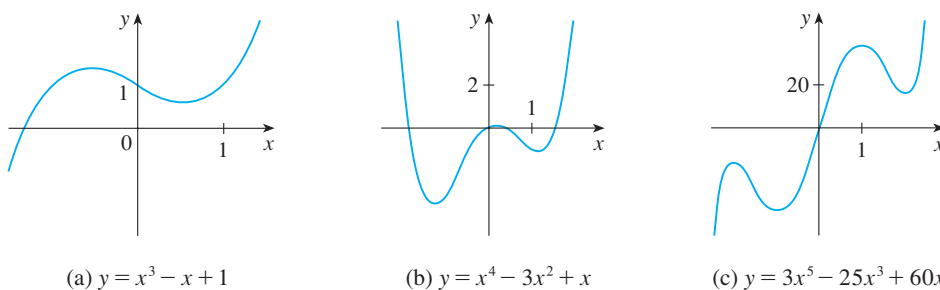


FIGURE 5

Polynomials are commonly used to model various quantities that occur in the natural and social sciences. For instance, in Chapter 2 we will explain why economists often use a polynomial $P(x)$ to represent the cost of producing x units of a commodity.

□ **POWER FUNCTIONS**

A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**. We consider several cases.

(i) $a = n$, where n is a positive integer

The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4,$ and 5 are shown in Figure 6. (These are polynomials with only one term.) You are familiar with the shape of the graphs of $y = x$ (a line through the origin with slope 1) and $y = x^2$ (a parabola).

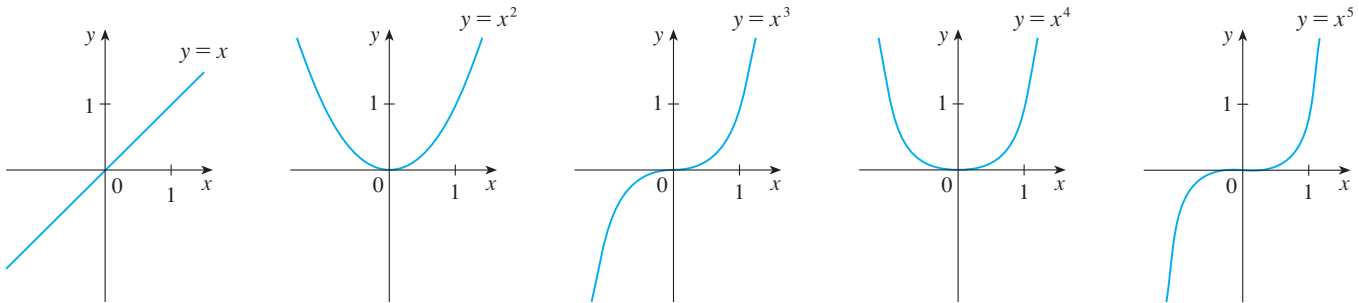


FIGURE 6 Graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5$

The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd. If n is even, then $f(x) = x^n$ is an even function and its graph is similar to the parabola $y = x^2$. If n is odd, then $f(x) = x^n$ is an odd function and its graph is similar to that of $y = x^3$. Notice from Figure 7, however, that as n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| \geq 1$. (If x is small, then x^2 is smaller, x^3 is even smaller, x^4 is smaller still, and so on.)

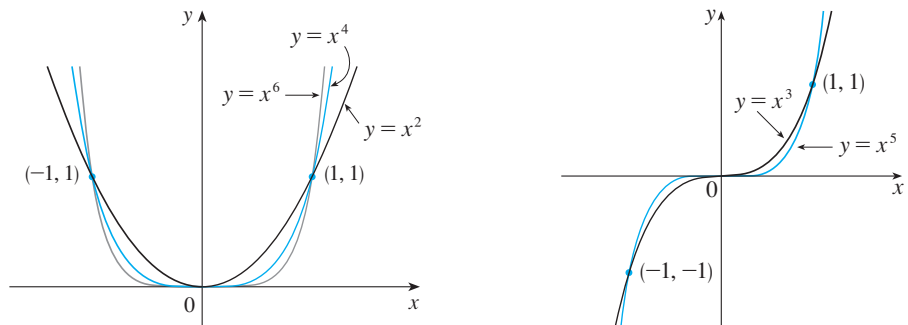


FIGURE 7
Families of power functions

(ii) $a = 1/n$, where n is a positive integer

The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**. For $n = 2$ it is the square root function $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$ and whose graph is the upper half of the parabola $x = y^2$. [See Figure 8(a).] For other even values of n , the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$. For $n = 3$ we have the cube root function

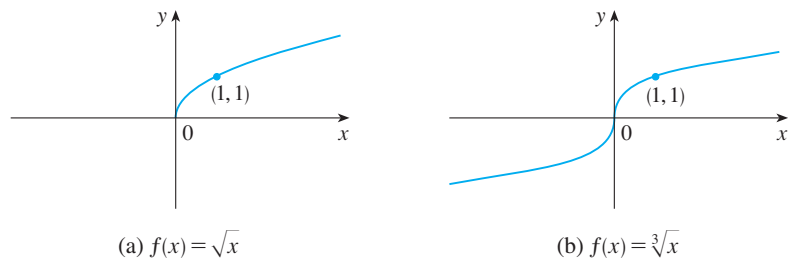


FIGURE 8
Graphs of root functions

(a) $f(x) = \sqrt{x}$

(b) $f(x) = \sqrt[3]{x}$

$f(x) = \sqrt[3]{x}$ whose domain is \mathbb{R} (recall that every real number has a cube root) and whose graph is shown in Figure 8(b). The graph of $y = \sqrt[n]{x}$ for n odd ($n > 3$) is similar to that of $y = \sqrt[3]{x}$.

(iii) $a = -1$

The graph of the **reciprocal function** $f(x) = x^{-1} = 1/x$ is shown in Figure 9. Its graph has the equation $y = 1/x$, or $xy = 1$, and is a hyperbola with the coordinate axes as its asymptotes. This function arises in physics and chemistry in connection with Boyle's Law, which says that, when the temperature is constant, the volume V of a gas is inversely proportional to the pressure P :

$$V = \frac{C}{P}$$

where C is a constant. Thus the graph of V as a function of P has the same general shape as the right half of Figure 9.

□ RATIONAL FUNCTIONS

A **rational function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain consists of all values of x such that $Q(x) \neq 0$. A simple example of a rational function is the function $f(x) = 1/x$, whose domain is $\{x \mid x \neq 0\}$; this is the reciprocal function graphed in Figure 9. The function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain $\{x \mid x \neq \pm 2\}$. Its graph is shown in Figure 10.

□ TRIGONOMETRIC FUNCTIONS

Trigonometry and the trigonometric functions are reviewed on Reference Page 2 and also in Appendix A. In calculus the convention is that radian measure is always used (except when otherwise indicated). For example, when we use the function $f(x) = \sin x$, it is understood that $\sin x$ means the sine of the angle whose radian measure is x . Thus the graphs of the sine and cosine functions are as shown in Figure 11.

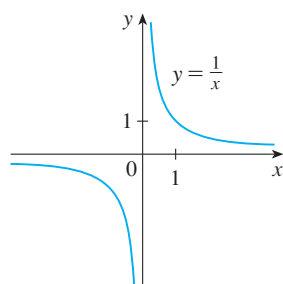


FIGURE 9
The reciprocal function

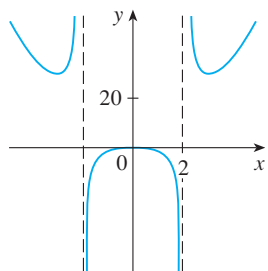
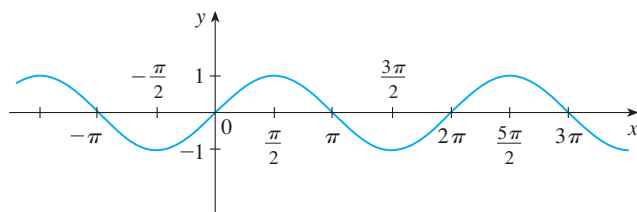
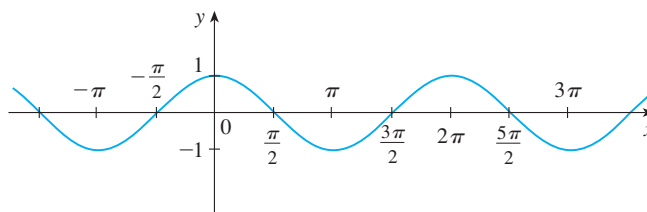


FIGURE 10
 $f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$



(a) $f(x) = \sin x$



(b) $g(x) = \cos x$

FIGURE 11

Notice that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x , we have

$$-1 \leq \sin x \leq 1 \quad -1 \leq \cos x \leq 1$$

or, in terms of absolute values,

$$|\sin x| \leq 1 \quad |\cos x| \leq 1$$

Also, the zeros of the sine function occur at the integer multiples of π ; that is,

$$\sin x = 0 \quad \text{when} \quad x = n\pi \quad n \text{ an integer}$$

An important property of the sine and cosine functions is that they are periodic functions and have period 2π . This means that, for all values of x ,

$$\sin(x + 2\pi) = \sin x \quad \cos(x + 2\pi) = \cos x$$

The periodic nature of these functions makes them suitable for modeling repetitive phenomena such as tides, vibrating springs, and sound waves.

The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

and its graph is shown in Figure 12. It is undefined whenever $\cos x = 0$, that is, when $x = \pm\pi/2, \pm3\pi/2, \dots$. Its range is $(-\infty, \infty)$. Notice that the tangent function has period π :

$$\tan(x + \pi) = \tan x \quad \text{for all } x$$

The remaining three trigonometric functions (cosecant, secant, and cotangent) are the reciprocals of the sine, cosine, and tangent functions. Their graphs are shown in Appendix A.

EXPONENTIAL FUNCTIONS AND LOGARITHMS

The **exponential functions** are the functions of the form $f(x) = a^x$, where the base a is a positive constant. The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown in Figure 13. In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

Exponential functions will be studied in detail in Section 3.1, and we will see that they are useful for modeling many natural phenomena, such as population growth (if $a > 1$) and radioactive decay (if $a < 1$).

The **logarithmic functions** $f(x) = \log_a x$, where the base a is a positive constant, are the inverse functions of the exponential functions. They will be studied in Section 3.2. Figure 14 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$.

TRANSFORMATIONS OF FUNCTIONS

By applying certain transformations to the graph of a given function we can obtain the graphs of certain related functions. This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given

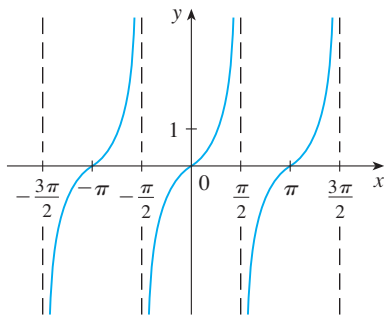


FIGURE 12
 $y = \tan x$

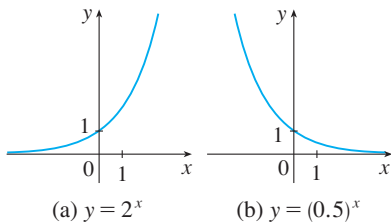


FIGURE 13

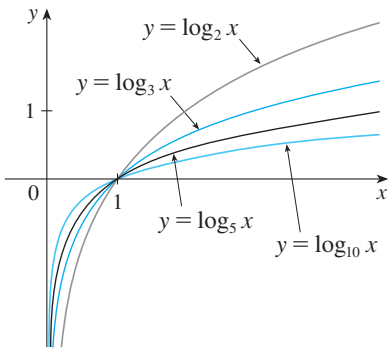


FIGURE 14

■ Figure 15 illustrates these shifts by showing how the graph of $y = (x + 3)^2 + 1$ is obtained from the graph of the parabola $y = x^2$: Shift 3 units to the left and 1 unit upward.

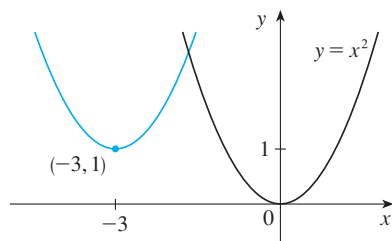


FIGURE 15
 $y = (x + 3)^2 + 1$

graphs. Let's first consider **translations**. If c is a positive number, then the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted upward a distance of c units (because each y -coordinate is increased by the same number c). Likewise, if $g(x) = f(x - c)$, where $c > 0$, then the value of g at x is the same as the value of f at $x - c$ (c units to the left of x). Therefore, the graph of $y = f(x - c)$ is just the graph of $y = f(x)$ shifted c units to the right.

VERTICAL AND HORIZONTAL SHIFTS Suppose $c > 0$. To obtain the graph of

- $y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
- $y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
- $y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
- $y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Now let's consider the **stretching** and **reflecting** transformations. If $c > 1$, then the graph of $y = cf(x)$ is the graph of $y = f(x)$ stretched by a factor of c in the vertical direction (because each y -coordinate is multiplied by the same number c). The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x -axis because the point (x, y) is replaced by the point $(x, -y)$. The following chart also incorporates the results of other stretching, compressing, and reflecting transformations.

VERTICAL AND HORIZONTAL STRETCHING AND REFLECTING

Suppose $c > 1$. To obtain the graph of

- $y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
- $y = (1/c)f(x)$, compress the graph of $y = f(x)$ vertically by a factor of c
- $y = f(cx)$, compress the graph of $y = f(x)$ horizontally by a factor of c
- $y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
- $y = -f(x)$, reflect the graph of $y = f(x)$ about the x -axis
- $y = f(-x)$, reflect the graph of $y = f(x)$ about the y -axis

Figure 16 illustrates these stretching transformations when applied to the cosine function with $c = 2$. For instance, in order to get the graph of $y = 2 \cos x$ we multiply the y -coordinate of each point on the graph of $y = \cos x$ by 2. This means that the graph of $y = \cos x$ gets stretched vertically by a factor of 2.

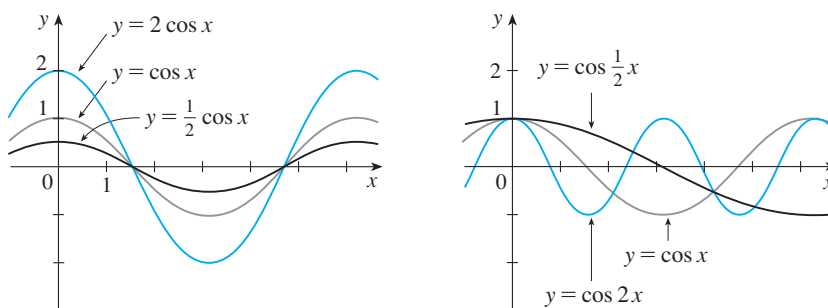


FIGURE 16

EXAMPLE 2 Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x - 2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

SOLUTION The graph of the square root function $y = \sqrt{x}$, obtained from Figure 8(a), is shown in Figure 17(a). In the other parts of the figure we sketch $y = \sqrt{x} - 2$ by shifting 2 units downward, $y = \sqrt{x - 2}$ by shifting 2 units to the right, $y = -\sqrt{x}$ by reflecting about the x -axis, $y = 2\sqrt{x}$ by stretching vertically by a factor of 2, and $y = \sqrt{-x}$ by reflecting about the y -axis.

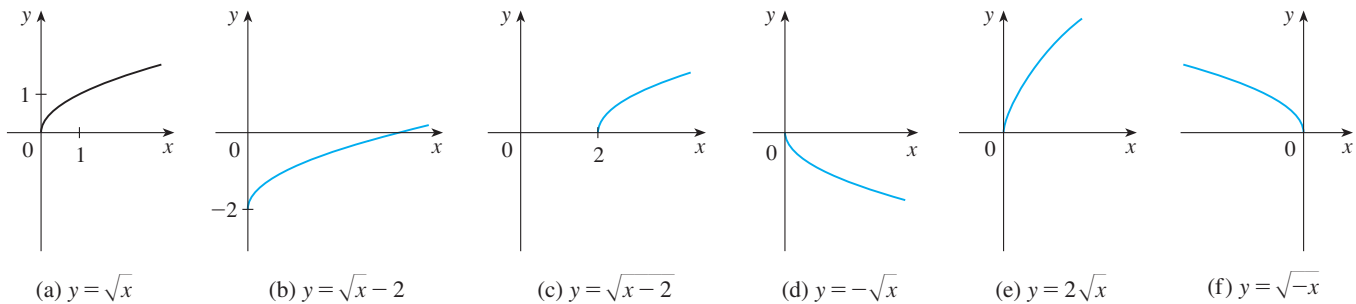


FIGURE 17

EXAMPLE 3 Sketch the graph of the function $y = 1 - \sin x$.

SOLUTION To obtain the graph of $y = 1 - \sin x$, we start with $y = \sin x$. We reflect about the x -axis to get the graph $y = -\sin x$ and then we shift 1 unit upward to get $y = 1 - \sin x$. (See Figure 18.)

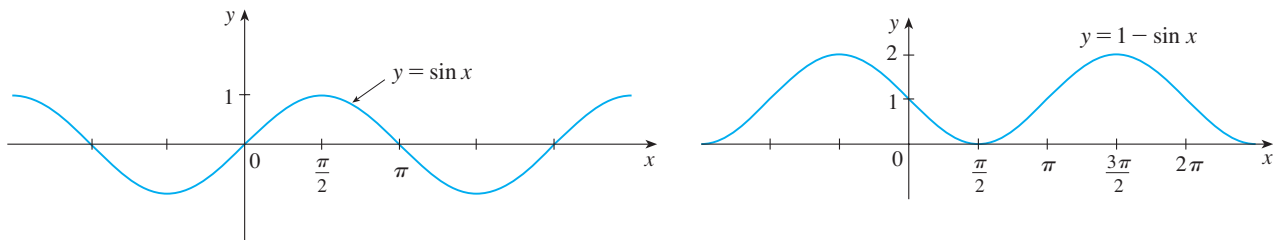


FIGURE 18

COMBINATIONS OF FUNCTIONS

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg , and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers. The sum and difference functions are defined by

$$(f + g)(x) = f(x) + g(x) \quad (f - g)(x) = f(x) - g(x)$$

If the domain of f is A and the domain of g is B , then the domain of $f + g$ is the intersection $A \cap B$ because both $f(x)$ and $g(x)$ have to be defined. For example, the domain of $f(x) = \sqrt{x}$ is $A = [0, \infty)$ and the domain of $g(x) = \sqrt{2 - x}$ is $B = (-\infty, 2]$, so the domain of $(f + g)(x) = \sqrt{x} + \sqrt{2 - x}$ is $A \cap B = [0, 2]$.

Similarly, the product and quotient functions are defined by

$$(fg)(x) = f(x)g(x) \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of fg is $A \cap B$, but we can't divide by 0 and so the domain of f/g is $\{x \in A \cap B \mid g(x) \neq 0\}$. For instance, if $f(x) = x^2$ and $g(x) = x - 1$, then the domain of the rational function $(f/g)(x) = x^2/(x - 1)$ is $\{x \mid x \neq 1\}$, or $(-\infty, 1) \cup (1, \infty)$.

There is another way of combining two functions to get a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$. Since y is a function of u and u is, in turn, a function of x , it follows that y is ultimately a function of x . We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g .

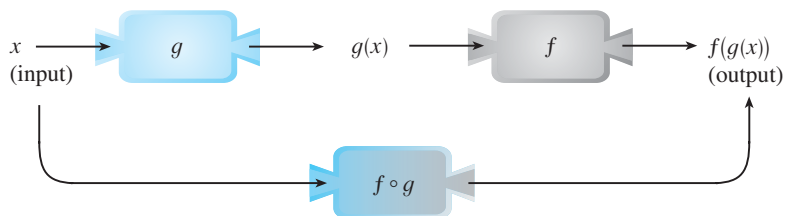
In general, given any two functions f and g , we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f , then we can calculate the value of $f(g(x))$. The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f . It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ (“ f circle g ”).

DEFINITION Given two functions f and g , the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f . In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined. Figure 19 shows how to picture $f \circ g$ in terms of machines.

FIGURE 19
The $f \circ g$ machine is composed of the g machine (first) and then the f machine.



EXAMPLE 4 If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

SOLUTION We have

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$$

NOTE You can see from Example 4 that, in general, $f \circ g \neq g \circ f$. Remember, the notation $f \circ g$ means that the function g is applied first and then f is applied second. In Example 4, $f \circ g$ is the function that *first* subtracts 3 and *then* squares; $g \circ f$ is the function that *first* squares and *then* subtracts 3.

EXAMPLE 5 If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$, find each function and its domain.

- (a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

SOLUTION

$$(a) \quad (f \circ g)(x) = f(g(x)) = f(\sqrt{2-x}) = \sqrt{\sqrt{2-x}} = \sqrt[4]{2-x}$$

The domain of $f \circ g$ is $\{x \mid 2-x \geq 0\} = \{x \mid x \leq 2\} = (-\infty, 2]$.

$$(b) \quad (g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{2-\sqrt{x}}$$

If $0 \leq a \leq b$, then $a^2 \leq b^2$.

For \sqrt{x} to be defined we must have $x \geq 0$. For $\sqrt{2-\sqrt{x}}$ to be defined we must have $2-\sqrt{x} \geq 0$, that is, $\sqrt{x} \leq 2$, or $x \leq 4$. Thus we have $0 \leq x \leq 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

$$(c) \quad (f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

The domain of $f \circ f$ is $[0, \infty)$.

$$(d) \quad (g \circ g)(x) = g(g(x)) = g(\sqrt{2-x}) = \sqrt{2-\sqrt{2-x}}$$

This expression is defined when both $2-x \geq 0$ and $2-\sqrt{2-x} \geq 0$. The first inequality means $x \leq 2$, and the second is equivalent to $\sqrt{2-x} \leq 2$, or $2-x \leq 4$, or $x \geq -2$. Thus, $-2 \leq x \leq 2$, so the domain of $g \circ g$ is the closed interval $[-2, 2]$. ■

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h , then g , and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to *decompose* a complicated function into simpler ones, as in the following example.

EXAMPLE 6 Given $F(x) = \cos^2(x+9)$, find functions f , g , and h such that $F = f \circ g \circ h$.

SOLUTION Since $F(x) = [\cos(x+9)]^2$, the formula for F says: First add 9, then take the cosine of the result, and finally square. So we let

$$h(x) = x + 9 \quad g(x) = \cos x \quad f(x) = x^2$$

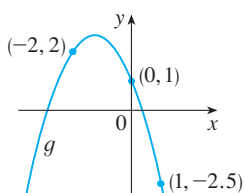
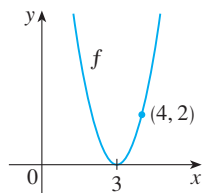
Then

$$\begin{aligned} (f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x+9)) = f(\cos(x+9)) \\ &= [\cos(x+9)]^2 = F(x) \end{aligned}$$

■

1.2 EXERCISES

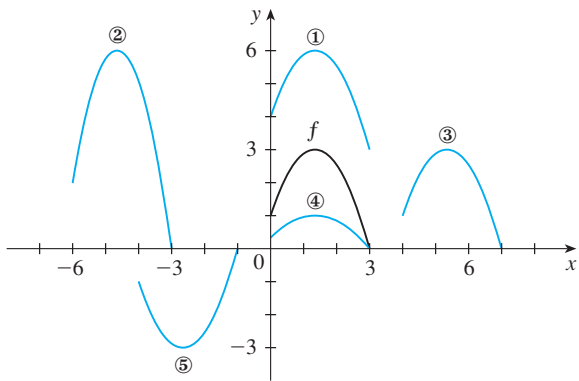
1. (a) Find an equation for the family of linear functions with slope 2 and sketch several members of the family.
 (b) Find an equation for the family of linear functions such that $f(2) = 1$ and sketch several members of the family.
 (c) Which function belongs to both families?
2. What do all members of the family of linear functions $f(x) = 1 + m(x + 3)$ have in common? Sketch several members of the family.
3. What do all members of the family of linear functions $f(x) = c - x$ have in common? Sketch several members of the family.
4. Find expressions for the quadratic functions whose graphs are shown.



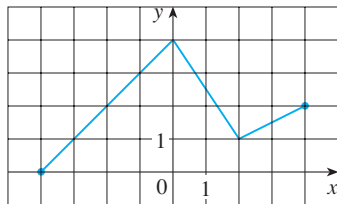
5. Find an expression for a cubic function f if $f(1) = 6$ and $f(-1) = f(0) = f(2) = 0$.
6. Some scientists believe that the average surface temperature of the world has been rising steadily. They have modeled the temperature by the linear function $T = 0.02t + 8.50$, where T is temperature in $^{\circ}\text{C}$ and t represents years since 1900.
 (a) What do the slope and T -intercept represent?
 (b) Use the equation to predict the average global surface temperature in 2100.
7. If the recommended adult dosage for a drug is D (in mg), then to determine the appropriate dosage c for a child of age a , pharmacists use the equation $c = 0.0417D(a + 1)$. Suppose the dosage for an adult is 200 mg.
 (a) Find the slope of the graph of c . What does it represent?
 (b) What is the dosage for a newborn?
8. The manager of a weekend flea market knows from past experience that if he charges x dollars for a rental space at the flea market, then the number y of spaces he can rent is given by the equation $y = 200 - 4x$.
 (a) Sketch a graph of this linear function. (Remember that the rental charge per space and the number of spaces rented can't be negative quantities.)
 (b) What do the slope, the y -intercept, and the x -intercept of the graph represent?
9. The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the linear function $F = \frac{9}{5}C + 32$.
 (a) Sketch a graph of this function.
 (b) What is the slope of the graph and what does it represent? What is the F -intercept and what does it represent?
10. Jason leaves Detroit at 2:00 PM and drives at a constant speed west along I-96. He passes Ann Arbor, 40 mi from Detroit, at 2:50 PM.
 (a) Express the distance traveled in terms of the time elapsed.
 (b) Draw the graph of the equation in part (a).
 (c) What is the slope of this line? What does it represent?
11. Biologists have noticed that the chirping rate of crickets of a certain species is related to temperature, and the relationship appears to be very nearly linear. A cricket produces 113 chirps per minute at 70°F and 173 chirps per minute at 80°F .
 (a) Find a linear equation that models the temperature T as a function of the number of chirps per minute N .
 (b) What is the slope of the graph? What does it represent?
 (c) If the crickets are chirping at 150 chirps per minute, estimate the temperature.
12. The manager of a furniture factory finds that it costs \$2200 to manufacture 100 chairs in one day and \$4800 to produce 300 chairs in one day.
 (a) Express the cost as a function of the number of chairs produced, assuming that it is linear. Then sketch the graph.
 (b) What is the slope of the graph and what does it represent?
 (c) What is the y -intercept of the graph and what does it represent?
13. At the surface of the ocean, the water pressure is the same as the air pressure above the water, 15 lb/in^2 . Below the surface, the water pressure increases by 4.34 lb/in^2 for every 10 ft of descent.
 (a) Express the water pressure as a function of the depth below the ocean surface.
 (b) At what depth is the pressure 100 lb/in^2 ?
14. The monthly cost of driving a car depends on the number of miles driven. Lynn found that in May it cost her \$380 to drive 480 mi and in June it cost her \$460 to drive 800 mi.
 (a) Express the monthly cost C as a function of the distance driven d , assuming that a linear relationship gives a suitable model.
 (b) Use part (a) to predict the cost of driving 1500 miles per month.
 (c) Draw the graph of the linear function. What does the slope represent?
 (d) What does the y -intercept represent?
 (e) Why does a linear function give a suitable model in this situation?

- 15.** Suppose the graph of f is given. Write equations for the graphs that are obtained from the graph of f as follows.
- (a) Shift 3 units upward.
 - (b) Shift 3 units downward.
 - (c) Shift 3 units to the right.
 - (d) Shift 3 units to the left.
 - (e) Reflect about the x -axis.
 - (f) Reflect about the y -axis.
 - (g) Stretch vertically by a factor of 3.
 - (h) Shrink vertically by a factor of 3.

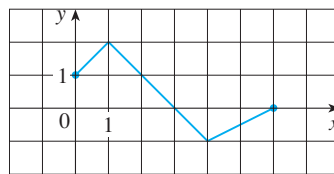
- 16.** Explain how the following graphs are obtained from the graph of $y = f(x)$.
- (a) $y = 5f(x)$
 - (b) $y = f(x - 5)$
 - (c) $y = -f(x)$
 - (d) $y = -5f(x)$
 - (e) $y = f(5x)$
 - (f) $y = 5f(x) - 3$
- 17.** The graph of $y = f(x)$ is given. Match each equation with its graph and give reasons for your choices.
- (a) $y = f(x - 4)$
 - (b) $y = f(x) + 3$
 - (c) $y = \frac{1}{3}f(x)$
 - (d) $y = -f(x + 4)$
 - (e) $y = 2f(x + 6)$



- 18.** The graph of f is given. Draw the graphs of the following functions.
- (a) $y = f(x + 4)$
 - (b) $y = f(x) + 4$
 - (c) $y = 2f(x)$
 - (d) $y = -\frac{1}{2}f(x) + 3$



- 19.** The graph of f is given. Use it to graph the following functions.
- (a) $y = f(2x)$
 - (b) $y = f(\frac{1}{2}x)$
 - (c) $y = f(-x)$
 - (d) $y = -f(-x)$



- 20.** (a) How is the graph of $y = 2 \sin x$ related to the graph of $y = \sin x$? Use your answer and Figure 18(a) to sketch the graph of $y = 2 \sin x$.
- (b) How is the graph of $y = 1 + \sqrt{x}$ related to the graph of $y = \sqrt{x}$? Use your answer and Figure 17(a) to sketch the graph of $y = 1 + \sqrt{x}$.

21–34 ■ Graph the function by hand, not by plotting points, but by starting with the graph of one of the standard functions and then applying the appropriate transformations.

- 21.** $y = -x^3$
- 22.** $y = 1 - x^2$
- 23.** $y = (x + 1)^2$
- 24.** $y = x^2 - 4x + 3$
- 25.** $y = 1 + 2 \cos x$
- 26.** $y = 4 \sin 3x$
- 27.** $y = \sin(x/2)$
- 28.** $y = \frac{1}{x - 4}$
- 29.** $y = \sqrt{x + 3}$
- 30.** $y = (x + 2)^4 + 3$
- 31.** $y = \frac{1}{2}(x^2 + 8x)$
- 32.** $y = 1 + \sqrt[3]{x - 1}$
- 33.** $y = \frac{2}{x + 1}$
- 34.** $y = \frac{1}{4} \tan\left(x - \frac{\pi}{4}\right)$

35–36 ■ Find $f + g$, $f - g$, fg , and f/g and state their domains.

- 35.** $f(x) = x^3 + 2x^2$, $g(x) = 3x^2 - 1$
- 36.** $f(x) = \sqrt{1 + x}$, $g(x) = \sqrt{1 - x}$

37–42 ■ Find the functions (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, and (d) $g \circ g$ and their domains.

- 37.** $f(x) = x^2 - 1$, $g(x) = 2x + 1$
- 38.** $f(x) = 1 - x^3$, $g(x) = 1/x$
- 39.** $f(x) = \sin x$, $g(x) = 1 - \sqrt{x}$
- 40.** $f(x) = 1 - 3x$, $g(x) = 5x^2 + 3x + 2$
- 41.** $f(x) = x + \frac{1}{x}$, $g(x) = \frac{x + 1}{x + 2}$
- 42.** $f(x) = \sqrt{2x + 3}$, $g(x) = x^2 + 1$

43–44 ■ Find $f \circ g \circ h$.

43. $f(x) = \sqrt{x-1}$, $g(x) = x^2 + 2$, $h(x) = x + 3$

44. $f(x) = \frac{2}{x+1}$, $g(x) = \cos x$, $h(x) = \sqrt{x+3}$

45–48 ■ Express the function in the form $f \circ g$.

45. $F(x) = (x^2 + 1)^{10}$ 46. $F(x) = \sin(\sqrt{x})$

47. $u(t) = \sqrt{\cos t}$ 48. $u(t) = \frac{\tan t}{1 + \tan t}$

49–51 ■ Express the function in the form $f \circ g \circ h$.

49. $H(x) = 1 - 3^{x^2}$ 50. $H(x) = \sqrt[3]{2 + |x|}$

51. $H(x) = \sec^4(\sqrt{x})$

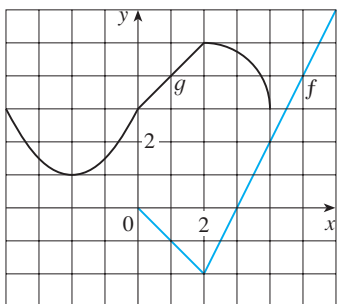
52. Use the table to evaluate each expression.

- (a) $f(g(1))$ (b) $g(f(1))$ (c) $f(f(1))$
 (d) $g(g(1))$ (e) $(g \circ f)(3)$ (f) $(f \circ g)(6)$

x	1	2	3	4	5	6
$f(x)$	3	1	4	2	2	5
$g(x)$	6	3	2	1	2	3

53. Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.

- (a) $f(g(2))$ (b) $g(f(0))$ (c) $(f \circ g)(0)$
 (d) $(g \circ f)(6)$ (e) $(g \circ g)(-2)$ (f) $(f \circ f)(4)$



54. A spherical balloon is being inflated and the radius of the balloon is increasing at a rate of 2 cm/s.

- (a) Express the radius r of the balloon as a function of the time t (in seconds).
 (b) If V is the volume of the balloon as a function of the radius, find $V \circ r$ and interpret it.

55. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.

- (a) Express the radius r of this circle as a function of the time t (in seconds).
 (b) If A is the area of this circle as a function of the radius, find $A \circ r$ and interpret it.

56. An airplane is flying at a speed of 350 mi/h at an altitude of one mile and passes directly over a radar station at time $t = 0$.

- (a) Express the horizontal distance d (in miles) that the plane has flown as a function of t .
 (b) Express the distance s between the plane and the radar station as a function of d .
 (c) Use composition to express s as a function of t .

57. The **Heaviside function** H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.

- (a) Sketch the graph of the Heaviside function.
 (b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and 120 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.
 (c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 5$ seconds and 240 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$. (Note that starting at $t = 5$ corresponds to a translation.)

58. The Heaviside function defined in Exercise 57 can also be used to define the **ramp function** $y = tH(t)$, which represents a gradual increase in voltage or current in a circuit.

- (a) Sketch the graph of the ramp function $y = tH(t)$.
 (b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and the voltage is gradually increased to 120 volts over a 60-second time interval. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 60$.
 (c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 7$ seconds and the voltage is gradually increased to 100 volts over a period of 25 seconds. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 32$.

59. Let f and g be linear functions with equations

$$f(x) = m_1x + b_1 \text{ and } g(x) = m_2x + b_2. \text{ Is } f \circ g \text{ also a linear function? If so, what is the slope of its graph?}$$

60. If you invest x dollars at 4% interest compounded annually, then the amount $A(x)$ of the investment after one year is $A(x) = 1.04x$. Find $A \circ A$, $A \circ A \circ A$, and $A \circ A \circ A \circ A$. What do these compositions represent? Find a formula for the composition of n copies of A .

61. (a) If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$. (Think about what operations you would have to perform on the formula for g to end up with the formula for h .)
 (b) If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.
62. If $f(x) = x + 4$ and $h(x) = 4x - 1$, find a function g such that $g \circ f = h$.
63. (a) Suppose f and g are even functions. What can you say about $f + g$ and fg ?
 (b) What if f and g are both odd?
64. Suppose f is even and g is odd. What can you say about fg ?
65. Suppose g is an even function and let $h = f \circ g$. Is h always an even function?
66. Suppose g is an odd function and let $h = f \circ g$. Is h always an odd function? What if f is odd? What if f is even?

1.3 THE LIMIT OF A FUNCTION

Our aim in this section is to explore the meaning of the limit of a function. We begin by showing how the idea of a limit arises when we try to find the velocity of a falling ball.

EXAMPLE 1 Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

SOLUTION Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.) If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = 4.9t^2$$

The difficulty in finding the velocity after 5 s is that we are dealing with a single instant of time ($t = 5$), so no time interval is involved. However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from $t = 5$ to $t = 5.1$:

$$\begin{aligned} \text{average velocity} &= \frac{\text{change in position}}{\text{time elapsed}} \\ &= \frac{s(5.1) - s(5)}{0.1} \\ &= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1} = 49.49 \text{ m/s} \end{aligned}$$

Time interval	Average velocity (m/s)
$5 \leq t \leq 6$	53.9
$5 \leq t \leq 5.1$	49.49
$5 \leq t \leq 5.05$	49.245
$5 \leq t \leq 5.01$	49.049
$5 \leq t \leq 5.001$	49.0049

The table shows the results of similar calculations of the average velocity over successively smaller time periods. It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The **instantaneous velocity** when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$. Thus the (instantaneous) velocity after 5 s is

$$v = 49 \text{ m/s}$$