Practice Problem Set 9

(1) Determine whether the given sequence $(a_n)_{n=1}^{\infty}$ converges or diverges with a_n as given. In the case of convergence, use the ε-N definition.

(a) $a_n := \dfrac{1}{n^p}$, for every $n \in \mathbb{N}$, where $p \in (0, \infty)$ if fixed.

(b) $a_n := \begin{cases} 0 & n \text{ odd}, \\ \dfrac{1}{n} & n \text{ even}, \end{cases}$ for every $n \in \mathbb{N}$.

(c) $a_n := \begin{cases} 1 & n \text{ odd}, \\ \dfrac{1}{n} & n \text{ even}, \end{cases}$ for every $n \in \mathbb{N}$.

(2) Use the ε-N definition for convergent sequences to show the following (*pay close attention to the general form/outline of your proof*):

(a) $\displaystyle\lim_{n\to\infty} \frac{2n+3}{n^3+1} = \frac{3}{2}$.

(3) Suppose that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are two sequences which differ from each other only in only a finite number of terms. Prove that $(a_n)_{n=1}^{\infty}$ converges to $L \in \mathbb{R}$ if and only if $(b_n)_{n=1}^{\infty}$ converges to $L \in \mathbb{R}$.

(4) Fix $A \in \mathbb{R}$ and suppose $(a_n)_{n=1}^{\infty}$ is a sequence such that $a_n = A$ for infinitely many values of $n \in \mathbb{N}$ (not necessarily consecutive terms).

(a) Show that if $(a_n)_{n=1}^{\infty}$ converges then $\displaystyle\lim_{n\to\infty} a_n = A$.

(b) Show that if $(a_n)_{n=1}^{\infty}$ diverges then $(a_n)_{n=1}^{\infty}$ does not necessarily need to converge to A.

(5) **TRUE OR FALSE:** If true, prove the statement. If false, provide a counter-example.

(a) If $(a_n)_{n=1}^{\infty}$ converges to 0, then given any positive number $\varepsilon \in (0, \infty)$, there exists $N \in \mathbb{N}$ such that $a_n < \varepsilon$ for all $n \in \mathbb{N}$, $n \geq N$.

(b) If for each $\varepsilon \in (0, \infty)$, there exists $N \in \mathbb{N}$ such that $a_n < \varepsilon$ for all $n \in \mathbb{N}$, $n \geq N$ then $(a_n)_{n=1}^{\infty}$ converges to 0.

(c) If $(a_n)_{n=1}^{\infty}$ converges to $A \in \mathbb{R}$ and $a_n > 0$ for every $n \in \mathbb{N}$ then $A > 0$.

(d) The sequence $\left((-1)^n \dfrac{n}{n+1}\right)_{n=1}^{\infty}$ converges.