Practice Problem Set 8

(1) Let $A \subseteq \mathbb{R}$ be a nonempty set which is bounded from below in $\mathbb{R}$ and suppose that $\beta \in \mathbb{R}$ is a lower bound for the set A. Prove that the following two statements are equivalent.

(a) $\beta = \inf(A)$.

(b) For every $\varepsilon \in (0, \infty)$ there exists $a \in A$ with the property that $a < \beta + \varepsilon$.

(2) Show that $\sup\big(\{1 - 1/n : n \in \mathbb{N}\}\big) = 1$.

(3) Suppose that $A \subseteq \mathbb{R}$ is a nonempty set which is bounded from below in $\mathbb{R}$ and fix a number $x \in \mathbb{R}$. If $x + A := \{x + a : a \in A\}$, then show that $\inf(x + A) = x + \inf(A)$.

(4) Suppose $a, b \in \mathbb{R}$.

(a) Show that $\max\{a, b\} = \dfrac{a + b + |a - b|}{2}$ and $\min\{a, b\} = \dfrac{a + b - |a - b|}{2}$.

Hint: Look at cases $a < b$, $a = b$, $a > b$.

(b) Use PMI to show that for every $n \in \mathbb{N}$, if $x_1, x_2, \ldots, x_n \in \mathbb{R}$ then $|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$.

(c) Show that if $x \in \mathbb{R}$ then $|x| = \sqrt{x^2}$.

(5) Let $x, y \in \mathbb{R}$. **TRUE OR FALSE:** If true prove it. If false, provide a counterexample.

(a) If $x < y$ then $|x| < |y|$.

(b) If $|x| < y$ then $|x^2| < y^2$.

(c) $|x| - |y| \leq |x - y|$.

(6) Show that a nonempty set $A \subseteq \mathbb{R}$ is bounded in $\mathbb{R}$ if and only if there exists $M \in \mathbb{R}$ such that $|a| \leq M$ for every $a \in A$.

(7) **TRUE OR FALSE:** Recall that the Nested Interval Property of $\mathbb{R}$ states that if $I_1, I_2, \cdots$ is a collection of nested closed intervals in $\mathbb{R}$ then there is a number $\xi \in \mathbb{R}$ such that $\xi \in I_n$, $\forall\, n \in \mathbb{N}$. Is the statement still true if each interval I_n is open?

(8) Suppose that $A, B \subseteq X$ are two nonempty sets which are bounded from below in X and that $A \subseteq B$. Show that if $\inf(A)$ and $\inf(B)$ exist in X then $\inf(B) \leq \inf(A)$.

(9) Show that the set of irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$, is uncountable.