Practice Problem Set 6

(1) Suppose that A and B are two sets. Show that

  (a) $A \cup B = A \cup (B \setminus A)$

  (b) $A \cap (B \setminus A) = \emptyset$

(2) If A, B, and C are sets and $B \subseteq C$ then $A \setminus C \subseteq A \setminus B$.

(3) Prove that if $A \subseteq X$ is a finite set then $A \cap B$ is finite for any set $B \subseteq X$.

(4) A set A is countably infinite if and only if there is a bijection $f : A \to \mathbb{N}$.

(5) A set A is countably infinite if and only if there is a countably infinite set B and a bijective mapping $f : A \to B$.

(6) For each part, give an example of two countably infinite sets A and B satisfying the desired property.

  (a) $A \cap B$ is countably infinite.

  (b) $A \setminus B$ is countably infinite.

  (c) $A \setminus B$ is finite and nonempty.

  Be sure to justify why your examples meet the criteria. For example, in (a), with your choices of A and B, justify why the set $A \cap B$ is countably infinite.

(7) Show that the union of finitely many countable sets is a countable set. That is, show that for each $n \in \mathbb{N}$, if $\{A_1, A_2, \ldots, A_n\}$ is a collection of countable sets, then the set $\bigcup_{k=1}^{n} A_k$ is countable.

(8) Let A be a set with n elements and B be a set with m elements for some $n, m \in \mathbb{N}$. Prove each of the following statements.

  (a) $n = m$ if and only if there is a bijection $\varphi : A \to B$.

  (b) $n \leq m$ if and only if there is an injection $\varphi : A \to B$.

  (c) $n \geq m$ if and only if there is a surjection $\varphi : A \to B$.

  (d) $n < m$ if and only if there is an injection $\varphi : A \to B$, but there is no surjection.

  (e) $n < m$ if and only if there is a surjection $\varphi : B \to A$, but there is no injection.

(9) Suppose that A and B are two finite sets both having n elements for some $n \in \mathbb{N}$.

  (a) Prove that any injective function $f : A \to B$ is also surjective. Show this is not necessarily true if A and B are infinite.

  (b) Prove that any surjective function $f : A \to B$ is also injective. Show this is not necessarily true if A and B are infinite.