Practice Problem Set 4

(1) Suppose A and B are sets and $f : A \to B$ is a given function. Prove the following statements.

(a) $S \subseteq f^{-1}(f(S)), \quad \forall\, S \subseteq A$.

(b) $f^{-1}(B \setminus S) = A \setminus f^{-1}(S) \quad \forall\, S \subseteq B$.

(c) If $C, D \subseteq A$ then $f(C \cup D) = f(C) \cup f(D)$.

(d) If $C, D \subseteq B$ then $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

(e) If $g : B \to C$ is another function, then

$$(g \circ f)^{-1}(D) = f^{-1}(g^{-1}(D)), \quad \forall\, D \subseteq C.$$

(f) For each $n \in \mathbb{N}$, if $C_1, \ldots, C_n \subseteq A$, then

$$f\left(\bigcup_{k=1}^{n} C_k\right) = \bigcup_{k=1}^{n} f(C_k).$$

(g) For each $n \in \mathbb{N}$, if $D_1, \ldots, D_n \subseteq B$, then

$$f^{-1}\left(\bigcap_{k=1}^{n} D_k\right) = \bigcap_{k=1}^{n} f^{-1}(D_k).$$

(h) If $y, z \in B$ with $y \neq z$, then $f^{-1}(\{y\}) \cap f^{-1}(\{z\}) = \emptyset$.

(2) Rewrite the following statements in the language of universal and/or existential quantifiers. The statements should limit the use of non-Mathematical English words. *Be sure to use the notations* $\forall$ *and* $\exists$. Do not worry about the truth value of the statement.

(a) For every real number x, there is a real number y for which $y^3 = x$.

(b) Given any two rational numbers a and b, it follows that the product ab is rational.

(3) Rewrite the following as statements as a sentence in English. You should limit the use of Mathematical notation. Do not worry about the truth value of the statement.

(a) $\forall\, x \in \mathbb{Q}$, $x^2 > 0$.

(b) $\forall\, x \in \mathbb{Q}$, $\exists n \in \mathbb{N}$, $x^n \geq 0$.

(4) Negate each of the following statements. It may be helpful to first rewrite the statements in the language of universal and/or existential quantifiers.

(a) The square of every real number is non-negative.

(b) For every real number x there is a real number y for which $y^3 = x$.