Practice Problem Set 3

Throughout this assignment you may use the basic algebraic properties of $\mathbb{Z}$ and $\mathbb{N}$ without proof.

(1) Prove that $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$, for each $n \in \mathbb{N}$.

(2) Show that for each $n \in \mathbb{N}$, there holds $(1+2+\cdots+n)^2 = 1^3 + 2^3 + \cdots + n^3$.
Hint: You may find it useful to use the fact that $1+2+\cdots+n = \frac{n(n+1)}{2}$.

(3) Find all values of $n \in \mathbb{N}$ for which the following statement is true:
If $A_1, A_2, \ldots, A_n$ and B are sets, then

$$B \bigcup \left(\bigcap_{k=1}^{n} A_k \right) = \bigcap_{k=1}^{n} (B \cup A_k).$$

Here, we employ the notation $\displaystyle\bigcap_{k=1}^{n} C_k = C_1 \cap C_2 \cap \cdots \cap C_n$, *if* $C_1, C_2, \ldots, C_n$ *are given sets.*

(4) (a) Show that $3n \leq 2^n$ for each $n \in \mathbb{N}$ with $n \geq 3$.
(b) Use part (a) to prove that $n^2 < 2^n$ for every $n \in \mathbb{N}$, with $n \geq 5$.

(5) Prove The Principle of Strong Induction stated in class.
Hint: Let $Q(n)$ be the statement "$P(1), P(2), \ldots, P(n)$ are all true" and use PMI to show that $Q(n)$ is true for all $n \in \mathbb{N}$.

(6) Suppose $x \in \mathbb{Z}$ is odd. Show that for each $n \in \mathbb{N}$ the integer x^n is odd.

(7) For each $n \in \mathbb{N}$, the integer $n^3 - n$ is divisible by 3.

(8) [**Towers of Hanoi**]
The towers of Hanoi is a game made famous by French mathematician Edouard Lucas in 1883. The game consists of three rods and a stack of disks. On the first rod there is a stack of disks arranged from largest (in diameter) on the bottom to smallest (in diameter) on top. The other two rods are empty. The object of the game is to transfer the entire stack of disks to one of the empty rods, obeying the following simple rules: 1. Only one disk can be moved at a time. 2. Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack. and 3. A larger disk never rests on top of a smaller disk. Use the Principle of Mathematical Induction to show that if you start with $n \in \mathbb{N}$ disks, then the game can be completed by performing $2^n - 1$ moves.

(9) Find the error in the "proof" of the following assertion:

Claim: *If $n \in \mathbb{N}$ and if $p, q \in \mathbb{N}$ are such that* $\max\{p, q\} = n$ *then* $p = q$.

"Proof" (By induction on n)
If $n = 1$ and if $p, q \in \mathbb{N}$ are such that $\max\{p, q\} = n$ then $p = q = 1$. Thus the claim is true for the base case. Next, fix $n \in \mathbb{N}$ and assume that the claim is true for n, i.e., assume that if $p, q \in \mathbb{N}$ are such that $\max\{p, q\} = n$ then $p = q$. We want to show that the claim is true for $n + 1$. As such, suppose that $p, q \in \mathbb{N}$ are such that $\max\{p, q\} = n + 1$. We need to show that $p = q$. Notice that $\max\{p-1, q-1\} = n$. Thus, by the induction hypothesis, we have that $p - 1 = q - 1$. Adding 1 to both sides of this last equality, we have $p = q$, as wanted. We can now conclude by PMI that the claim is true for all $n \in \mathbb{N}$. □