Practice Problem Set 10

(1) Let $A \subseteq \mathbb{R}$ be a nonempty set which is bounded from below in $\mathbb{R}$ (so that $\inf(A) \in \mathbb{R}$ exists). Prove that there exist a nonincreasing sequence $(y_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} y_n = \inf(A)$.

(2) Fix a set $E \subseteq \mathbb{R}$ and suppose that $(a_n)_{n=1}^{\infty}$ such that $a_n \in E$ for every $n \in \mathbb{N}$. Show that if $A, B \subseteq E$ and $E = A \cup B$ then either A or B contains infinitely many terms of $(a_n)_{n=1}^{\infty}$.

(3) Give an example of sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that $(a_n)_{n=1}^{\infty}$ converges to 0 but $(a_n b_n)_{n=1}^{\infty}$ diverges.

(4) (a) Given an example of two divergent sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that $(a_n + b_n)_{n=1}^{\infty}$ converges.

(b) Prove that if $(a_n)_{n=1}^{\infty}$ converges and $(b_n)_{n=1}^{\infty}$ diverges then the sum $(a_n + b_n)_{n=1}^{\infty}$ diverges.

(c) Given an example of two divergent sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that $(a_n b_n)_{n=1}^{\infty}$ converges.

(d) Prove that if $(a_n)_{n=1}^{\infty}$ converges to a nonzero number and $(b_n)_{n=1}^{\infty}$ diverges then the product $(a_n b_n)_{n=1}^{\infty}$ diverges.

(5) Suppose that the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ both converge to $A \in \mathbb{R}$.

(a) The sequence $(z_n)_{n=1}^{\infty} = (a_1, b_1, a_2, b_2, a_3, b_3, \dots)$ is called a *zipper sequence*. Find a formula for the nth term z_n.

(b) Does $(z_n)_{n=1}^{\infty}$ converge? If so, prove your claim using the ε-N definition. If not, provide an example of sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ which both converge to $A \in \mathbb{R}$ but $(z_n)_{n=1}^{\infty}$ diverges.

(6) Let $(a_n)_{n=1}^{\infty}$ be a sequence and suppose that $(a_{n_k})_{k=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$. Use induction to show that $k \le n_k$ for every $k \in \mathbb{N}$.

(7) Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence where $a_n \ge 0$ for every $n \in \mathbb{N}$. Prove that if the sequence $\left((-1)^n a_n\right)_{n=1}^{\infty}$ converges then $(a_n)_{n=1}^{\infty}$ converges. Where, if anywhere, did you need the assumption that $a_n \ge 0$?

(8) Determine if each recursively defined sequence $(a_n)_{n=1}^{\infty}$ converges or diverges. In the case of convergence, find its limit.

(a) $a_1 = 2$ and $a_{n+1} = 1/(a_n)^2$ for all $n \in \mathbb{N}$.

(9) Prove that if $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence whose terms are integers then it is eventually a constant sequence.

(10) Prove that every unbounded sequence contains a monotone subsequence.

(11) Using the Cauchy sequence definition, show that each sequence $(a_n)_{n=1}^{\infty}$ (as given) is Cauchy.

(a) $a_n = \dfrac{n}{n+1}$, $n \in \mathbb{N}$.

(b) $a_n = \dfrac{1}{\sqrt{n}}$, $n \in \mathbb{N}$.