HOMEWORK VII

Due: Thursday, November 9th, 2017.

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1. Let $A := \left\{ \frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N} \right\}$. Find the values of $\sup(A)$ and $\inf(A)$.
2. Let $A \subseteq X$ be a nonempty subset such that $\sup(A) \in X$ exists. Suppose that there is a set $B \subseteq A$ with the property that whenever $x \in A$ there is a $y \in B$ such that $x \leq y$. Show that $\sup(B) \in X$ exists and $\sup(B) = \sup(A)$.
3. Let $A \subseteq X$ be a nonempty set that is bounded from above. Suppose that $\sup(A)$ exists and that $\sup(A) \notin A$. Show that A contains a countably infinite subset (thus, A is an infinite set).
4. Suppose that $A, B \subseteq \mathbb{R}$ are two nonempty sets which are bounded from below in $\mathbb{R}$. Show that $\inf(A + B) = \inf(A) + \inf(B)$.
5. Suppose $A \subseteq (0, \infty)$ is nonempty and bounded from above and fix a number $\lambda \in (0, \infty)$. With $A^\lambda := \{a^\lambda : a \in A\}$, show that $\sup\left(A^\lambda\right) = \left[\sup(A)\right]^\lambda$. Give an example of a set $A \subseteq \mathbb{R}$ (which is nonempty and bounded from above) and an exponent $\lambda \in (0, \infty)$ which shows the conclusion may be false if $A \not\subseteq (0, \infty)$.
6. Suppose that $A \subseteq \mathbb{R}$ is a nonempty set which is bounded from below in $\mathbb{R}$.
   (a) Show that if $\lambda \in \mathbb{R}$ with $\lambda < 0$, then $\sup(\lambda A) = \lambda \inf(A)$.
   (b) Show that the Least Upper Bound Property of $\mathbb{R}$ implies that $\mathbb{R}$ has the Greatest Lower Bound Property in the sense that whenever $A \subseteq \mathbb{R}$ is a nonempty set which is bounded from below, it follows that $\inf(A) \in \mathbb{R}$ exists.
   Hint: Show that the set $-A := \{-a : a \in A\} \subseteq \mathbb{R}$ *is nonempty and bounded from above in* $\mathbb{R}$*. Then use the Least Upper Bound Property of* $\mathbb{R}$ *and part (a) of this question to show that* $\inf(A) \in \mathbb{R}$ *exists.*