HOMEWORK V

Due: Thursday, October 12, 2017.

---

1. Use PMI to show that the union of finitely many finite sets is a finite set, i.e., show that for each $n \in \mathbb{N}$, if $\{A_1, A_2, \ldots, A_n\}$ is a collection of finite sets, then the set $\bigcup_{k=1}^{n} A_k$ is finite.

   Be careful: these sets are not necessarily disjoint.

2. For each $n \in \mathbb{N}$ suppose that A_n is a nonempty finite set. Prove or disprove (with a counterexample): The union $\bigcup_{n=1}^{\infty} A_n$ is an infinite set.

3. Suppose that A is a finite set. Prove the if B is an infinite set then the set $B \setminus A$ is infinite.

4. Given an infinite set A, show that if $f : A \to B$ is an injective function for some set B, then $f(A)$ is an infinite set.

5. Let A be a set with n elements and B be a set with m elements for some $n, m \in \mathbb{N}$. Prove that $n \leq m$ if and only if there is an injection $\varphi : A \to B$.

6. Suppose that A and B are two finite sets both having n elements for some $n \in \mathbb{N}$. Show that any injective function $f : A \to B$ is also surjective. Show this is not necessarily true if A and B are infinite.

7. Show that if A is a countable set and B is a finite set, then the union $A \cup B$ is countable.

   Hint: Try constructing a piecewise function.

8. Suppose that A, B, C, and D are sets and that $f : A \to B$ and $g : C \to D$ are two bijective functions. Show that the function $h : A \times C \to B \times D$ defined by setting

   $$h(x, y) := \big(f(x), g(y)\big), \quad \text{for every } (x, y) \in A \times C,$$

   is bijective.

9. Show that if A and B are two countable sets then the Cartesian product $A \times B$ is countable.

   Hint: Use the previous problem and the fact that $\mathbb{N} \times \mathbb{N}$ *is countable.*