HOMEWORK III

Due: Tuesday, September 26, 2017.

---

1. Suppose A and B are sets and $f : A \to B$ is a given function. Prove the following statements.
   (a) $f(f^{-1}(S)) \subseteq S, \quad \forall S \subseteq B$. Show that one has equality if f is surjective.
   (b) If $C, D \subseteq B$ then $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.
   (c) If $C, D \subseteq A$ then $f(C \cap D) \subseteq f(C) \cap f(D)$. Show that one has equality if f is injective.
   (d) $f(A) \setminus f(S) \subseteq f(A \setminus S), \quad \forall S \subseteq A$. Show that one has equality if f is surjective.
   (e) For each $n \in \mathbb{N}$, if $D_1, \ldots, D_n \subseteq B$, then
   $$f^{-1}\left(\bigcup_{k=1}^{n} D_k\right) = \bigcup_{k=1}^{n} f^{-1}(D_k).$$

2. Rewrite the following statement into a quantified conditional statement.

   "Every integer that is not odd is even."

3. Rewrite the following statements in the language of universal and/or existential quantifiers. *Be sure to use the notations* $\forall$ *and* $\exists$. Do not worry about the truth value of the statement.

   "If $x \in [0, 1]$ then $x + y^2 = 1$ for some $y \in [-1, 1]$."

4. Negate each of the following statements. It may be helpful to first rewrite the statements in the language of universal and/or existential quantifiers.
   (a) For every strictly positive number ε, there is a strictly positive number M for which there holds $|f(x) - b| < \varepsilon$ whenever $x > M$.
   (b) If x is prime, then $\sqrt{x}$ is not a rational number.

5. Find the error in the "proof" of the following assertion:

   Claim: *Given* $a \in \mathbb{R}$, *one has that* $a^n = 1$ *for all* $n \in \mathbb{N} \cup \{0\}$.

   "Proof" (By strong induction on n)

   If $n = 0$ then $a^0 = 1$. Thus the claim is true for the base case. Next, fix $n \in \mathbb{N} \cup \{0\}$ and assume that the claim is true for all $k \in \mathbb{N} \cup \{0\}$ with $k \leq n$, i.e., assume that $a^k = 1$ for all $k \in \mathbb{N} \cup \{0\}$ with $k \leq n$. We want to show that the claim is true for $n + 1$. Observe that
   $$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1,$$
   where we have used the induction hypothesis in the second equality. Thus the claim is true for $n + 1$ and by PMI we can now conclude that the claim is true for all $n \in \mathbb{N} \cup \{0\}$. $\square$