HOMEWORK II

Due: Tuesday, September 19, 2017.

Throughout this assignment you may use algebraic properties of $\mathbb{Z}$ and $\mathbb{N}$ without proof.

1. Suppose that $n, m \in \mathbb{Z}$. Prove the following statements.
   (a) There is no rational solution to the equation $x^2 = 10$.
   (b) If $n^2(m+3)$ is even then n is even or m is odd.
   (c) If $n+m$ is even then either n and m are both even or both odd.
   (d) If $n+m$ is odd then either n is odd or m is odd.
   (e) $n^2 - 4m \neq 2$.

2. Prove that $\sum_{k=1}^{n}(-1)^k k^2 = (-1)^n \frac{n(n+1)}{2}$, for each $n \in \mathbb{N}$.

3. Prove that DeMorgan's Laws are true for finite collections of sets, i.e., show that if $n \in \mathbb{N}$ and $A_1, A_2, \ldots, A_n \subseteq X$ are sets then

$$\left(\bigcap_{k=1}^{n} A_k\right)^c = \bigcup_{k=1}^{n} A_k^c \quad \text{and} \quad \left(\bigcup_{k=1}^{n} A_k\right)^c = \bigcap_{k=1}^{n} A_k^c.$$

   Here, we employ the notation $\bigcap_{k=1}^{n} A_k = A_1 \cap A_2 \cap \cdots \cap A_n$ *and* $\bigcup_{k=1}^{n} A_k = A_1 \cup A_2 \cup \cdots \cup A_n$.

4. Prove the following statement: For each $n \in \mathbb{N}$, if $x_1, x_2, \ldots, x_n$ are all odd integers, then the product $x_1 x_2 \cdots x_n$ is odd.

5. (a) Suppose $x, y \in \mathbb{N}$ are such that $x \leq y$. Show that $x^n \leq y^n$ for every $n \in \mathbb{N}$.
   (b) Use part (a) to prove that $2^n + 1 \leq 3^n$ for every $n \in \mathbb{N}$.

6. Prove that every integer greater than or equal to 2 is divisible by at least one prime number.

   Do not use the Fundamental Theorem of Arithmetic to prove this result.